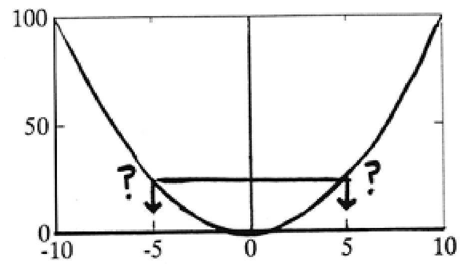
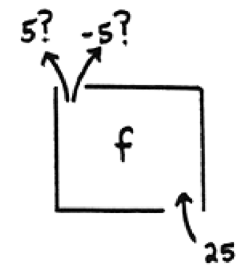
2.4 One-to-One Functions and Inverse Functions

Introduction

Recall that a *function* satisfies the property that for every input there exists a *unique* output. Thus, the graph of a function must pass a vertical line test.

Note what this definition does *not* say: it does not say that every *output* must be associated with a unique *input*. To illustrate this idea, consider the function f given by the rule $f(x) = x^2$:





When 5 is the input, $f(5) = 25$ is the output; and when -5 is the input, f(-5) = 25 is again the output. In terms of a 'black box', if we were to try to use the box 'backwards' and put the output 25 in the bottom, to see what input led to that output, the box would object: it doesn't know which input to choose. From a graphical point of view, functions need not pass a horizontal line test.

one-to-one functions; informal discussion

If the graph of a given function does indeed pass a horizontal line test, then this function has the additional property that for every *output*, there is a unique *input*. In this case, the function is given a special adjective: it is called a *one-to-one* function (abbreviated as '1-1'). The name is completely appropriate, for in this case there is a *one-to-one correspondence* between the outputs and the inputs: for every input, there exists a unique output (the 'function' condition), AND for every output, there exists a unique input (the '1-1' condition).

Observe that every *1-1 function* is firstly a *function*; we do not talk about the 1-1 property for non-functions.

In terms of a 'black box', the 1-1 property can be described as follows: we can stick an *output* in the bottom, and say, without ambiguity, what *input* must have been put in to produce this output. This is an extremely nice relationship between inputs and outputs: the inputs uniquely identify the outputs, *and* the outputs uniquely identify the inputs.

some symbols:
$\forall$*, 'for all'*
$\exists$*, 'there exists'*
!*, 'a unique'*

The following phrases occur so frequently in mathematics that there are special symbols for them:

The symbol ' $\forall$ ' is read as 'for all' or 'for every'.

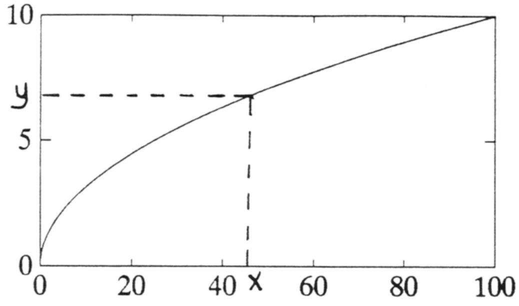
The symbol ' $\exists$ ' is read as 'there exists'.

The symbol ' ! ' is read as 'a unique'.

These symbols will be used freely throughout the text.

EXAMPLE
a 1-1 function

The function f given by $f(x) = \sqrt{x}$ is a one-to-one function. It is a *function* because for every $x \in \mathcal{D}(f)$, there exists a unique $y \in \mathcal{R}(f)$. Thus the graph of f passes the vertical line test. It is *one-to-one* because for every $y \in \mathcal{R}(f)$, there exists a unique $x \in \mathcal{D}(f)$. Thus, the graph of f passes the horizontal line test.



A 1-1 FUNCTION:

$\forall\ x \in \mathcal{D}(f),\ \exists\,!\ y \in \mathcal{R}(f)$

and

$\forall\ y \in \mathcal{R}(f),\ \exists\,!\ x \in \mathcal{D}(f)$

using the symbols $\forall$, $\exists$, !

The properties discussed in the previous paragraph can be stated concisely using the special symbols provided by the language of mathematics. For example, the 'function' condition

$$\forall\ x \in \mathcal{D}(f), \quad \exists\,!\ y \in \mathcal{R}(f)$$

is read as: *for all x in the domain of f, there exists a unique y in the range of f.* If this sentence were to appear in text, instead of being displayed (that is, set off and centered), then it would begin with the *words* 'For all' instead of the *symbol* $\forall$. It is not good style for a text sentence to begin with a symbol. However, in display mode, it is acceptable to begin a sentence with a symbol.

EXERCISE 1

♣ Write the following sentence using appropriate symbols: *For every y in the range of a function f, there exists a unique x in the domain of f.* What condition is being described here?

EXERCISE 2

♣ 1. Draw a graph of a non-function.

♣ 2. Draw a graph of a function that *is not* 1-1.

♣ 3. Draw a graph of a function that *is* 1-1.

♣ 4. Draw the graph of a relation between x and y such that y is a function of x and x is a function of y. Can you use a previous example?

♣ 5. Draw the graph of a relation between x and y such that y is a function of x but x is not a function of y. Can you use a previous example?

♣ 6. Is it possible to be a *one-to-one function* without being a *function*? Why or why not?

★★

precise definition of a 1-1 function

Here's the precise definition of a 1-1 function:

Let f be a function. Then:

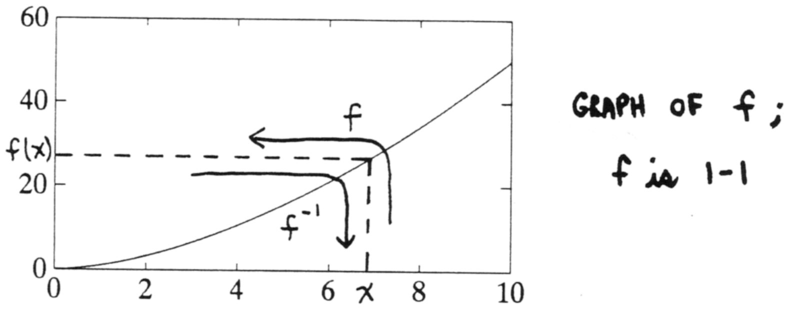
$$f \text{ is 1-1} \iff \forall\, x, y \in \mathcal{D}(f),\ f(x) = f(y) \implies x = y$$

This definition says: Whenever two outputs are the same, then the corresponding inputs must be the same. This is the form most often used if one is asked to *prove* that a function is 1-1.

However, this definition requires an understanding of the mathematical sentence called an *implication*; that is, a sentence of the form $A \Rightarrow B$. Implications are discussed in a future section.

inverse function f^{-1}

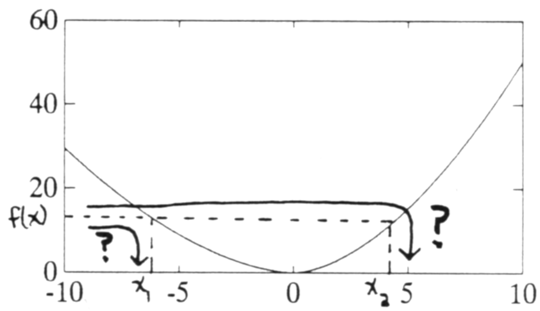
Whenever a function f is 1-1, then another function, called f^{-1} (read as '*f inverse*') can be defined that 'undoes' what f does! The picture below illustrates this fact:



Given an input x, the function f sends it to the (unique) output $f(x)$.

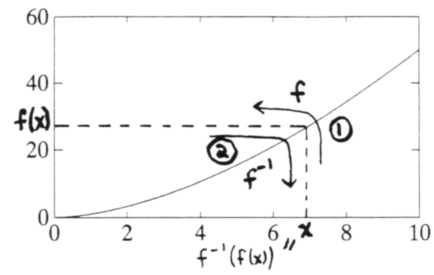
Given this output $f(x)$, the function f^{-1} sends it back to the (unique) input x.

Note: if f is NOT 1-1, then we can't do this, as illustrated below.

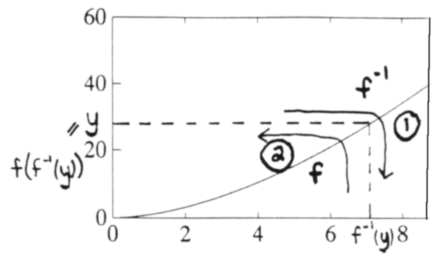


relationship between a function f and its inverse f^{-1}

For a 1-1 function f, the relationship between f and its inverse f^{-1} can be summarized via the sketches below. Look at the graph on the left. Begin with an x in the domain of f. The function f sends this input to $f(x)$. Then, the function f^{-1} takes $f(x)$ as its input, and sends us to the output $f^{-1}(f(x))$. But this output is just where we started: $f^{-1}(f(x)) = x$!



$\forall\ x \in \mathcal{D}(f),\ f^{-1}(f(x)) = x$



$\forall\ y \in \mathcal{R}(f),\ f(f^{-1}(y)) = y$

EXERCISE 3	♣ Look at the right-most preceding graph, labeled $\forall\ y \in \mathcal{R}(f),\ f\big(f^{-1}(y)\big) = y$. Explain, in words, what is happening here. Why do you suppose the dummy variable 'y' was used to represent a typical element of the range of f?

EXAMPLE

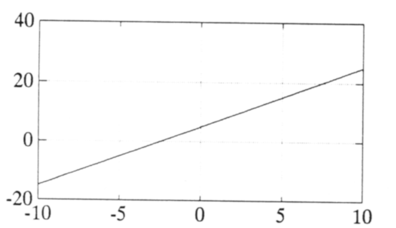
finding f^{-1}

Problem: Consider the function f given by the rule $f(x) = 2x + 5$. Show that f is 1-1. Then, find f^{-1}.

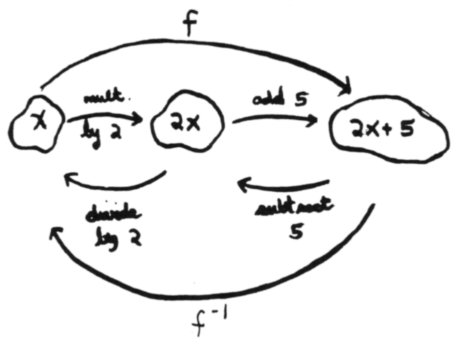
Solution: A quick graph of f shows that it passes both vertical and horizontal line tests, and hence is 1-1.

Now, let's find its inverse, in two different ways.

Method I (mapping approach): What does f DO? Well, it takes an input, multiplies it by 2, then adds 5. How could we UNDO this process? Just work backwards, 'reversing' the operations: first, subtract 5; then, divide by 2. Thus, $f^{-1}(x) = \frac{x-5}{2}$.



Graph of f,
f(x) = 2x + 5



Method II (algebraic approach): To find f^{-1}, use the fact that f and f^{-1} must satisfy the relationship $f(f^{-1}(x)) = x$. Treat the output $f^{-1}(x)$ as the 'unknown', and solve for it!

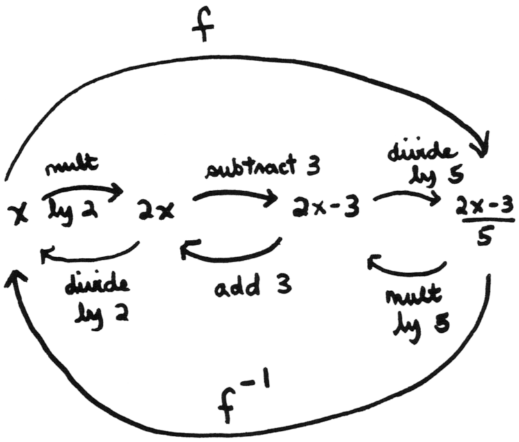
$$
\begin{aligned}
f(f^{-1}(x)) &= x && \text{(f and f^{-1} must satisfy this)} \\
2\big(f^{-1}(x)\big) + 5 &= x && \text{(definition of f)} \\
f^{-1}(x) &= \frac{x-5}{2} && \text{(solve for $f^{-1}(x)$)}
\end{aligned}
$$

EXAMPLE
finding f^{-1}

Problem: Find the inverse function for $f(x) = \frac{2x-3}{5}$ by thinking about *undoing* what f does. Then check your answer by verifying that both $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$ (for appropriate x).

Solution: Observe that f takes an input, multiplies it by 2, subtracts 3, then divides by 5. To undo this, f^{-1} must take its input, multiply by 5, add 3, and divide by 2, so that $f^{-1}(x) = \frac{5x+3}{2}$.

To check, observe that:



$$
\begin{aligned}
f^{-1}(f(x)) &= f^{-1}\big(\frac{2x-3}{5}\big) \\
&= \frac{5(\frac{2x-3}{5}) + 3}{2} = \frac{2x-3+3}{2} \\
&= x
\end{aligned}
$$

and

$$
\begin{aligned}
f(f^{-1}(x)) &= f\left(\frac{5x+3}{2}\right) \\
&= \frac{2\big(\frac{5x+3}{2}\big) - 3}{5} \\
&= \frac{5x+3-3}{5} \\
&= x
\end{aligned}
$$

EXERCISE 4

The following functions are $1-1$. (Why?) Find the inverse functions in two different ways: using a 'mapping approach', and an 'algebraic approach'. Verify the two conditions: $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

♣ 1. $f(x) = 4x + 1$

♣ 2. $g(x) = \frac{3-5x}{7}$

relationship between the graphs of f and f^{-1}

The points on the graph of f are of the form:

$$\{(x, f(x)) \mid x \in \mathcal{D}(f)\}$$

The points on the graph of f^{-1} are of the form:

$$\{(f(x), x) \mid x \in \mathcal{D}(f)\}$$

That is, whenever a point (a, b) is on the graph of f, there is a point (b, a) on the graph of f^{-1}. This situation leads to a nice relationship between the graphs of f and f^{-1}, which is called *symmetry about the line* $y = x$, and is discussed next.

symmetry about the line $y = x$

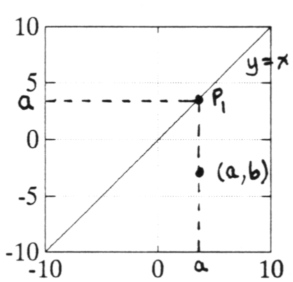
Graph the line $y = x$, and then plot a point (a, b) not on this line. Let's *use* the line $y = x$ to 'find' the point (b, a), as described below.

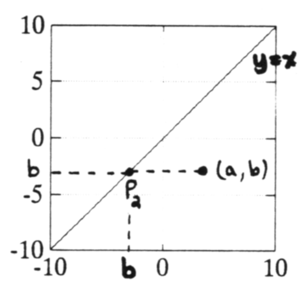
Refer to the left-most sketch below. First, go vertically from the point (a, b) to the line $y = x$, arriving at the point labeled P_1. The x-coordinate of P_1 is the same as that of (a, b) (why?), so P_1 must have coordinates $(a, ???)$. But, P_1 lies on the line $y = x$, so its x and y coordinates are the same. Thus, P_1 must be the point (a, a). Label the y-coordinate of P_1.

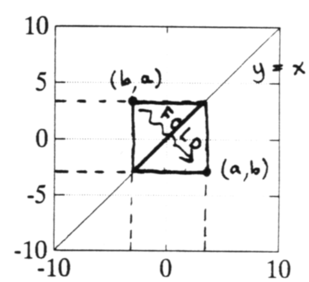
Now refer to the middle sketch below. Go horizontally from the point (a, b) to the line $y = x$, arriving at the point labeled P_2. This point P_2 has the same y-coordinate as (a, b) (why?), and hence P_2 is of the form $(???, b)$. But P_2 lies on the line $y = x$, so its x and y coordinates are the same. Thus, P_2 must be the point (b, b). Label the x-coordinate of P_2.

Look at the right-most sketch below. We used (a, b) and the line $y = x$ to find (b, a). A square is formed (with side of length $a - b$). The line $y = x$ forms a diagonal of this square. Thus, if we were to 'fold' the graph along the line $y = x$, the points would land on top of each other.

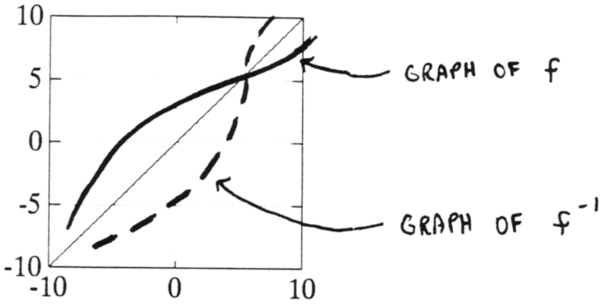
In other words, if you have a point (a, b) and WANT the point (b, a), just 'fold' along the line $y = x$! Curves that have the property that they lie atop each other, when folded along the line $y = x$, are said to be *symmetric about the line* $y = x$.





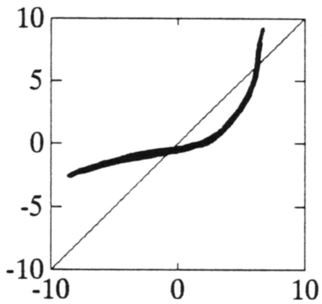


Now, given the graph of f, it is easy to find the graph of f^{-1}. Just 'fold' along the line $y = x$ (mentally), and sketch in the resulting curve, as illustrated below.



EXERCISE 5

♣ 1. On the graph below, sketch in a curve that is symmetric to the one drawn, about the line $y = x$.



♣ 2. Convince yourself that $f(x) = x^3$ is $1-1$. Then, find f^{-1}. Finally, graph both f and f^{-1} on the same graph.

a calculator exercise

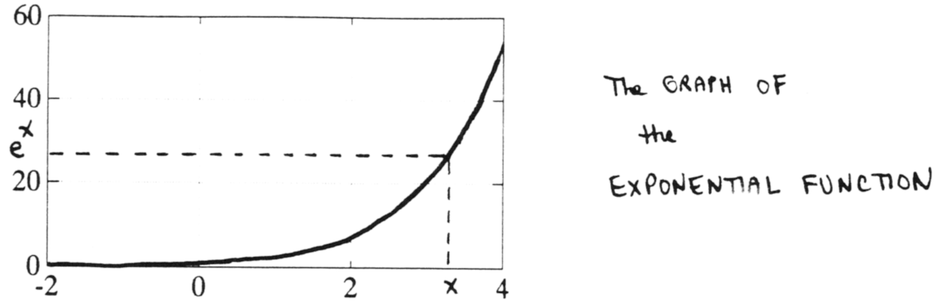
Pull out your hand-held calculator. Find keys that are labeled 'e^x' and 'ln'. (Ask you instructor for help, if necessary. Occasionally they are labeled slightly differently.) Now, try the following exercise:

`Input the number 2.`
`Press the` e^x `key. (Something like 7.39 will be displayed.)`
`Press the 'ln' key. (2 is displayed again.)`

Repeat this exercise, starting with 4.2 instead of 2.
Then repeat again, starting with -7.02.
What's happening here?

the exponential function; e^x

The key labeled e^x gives you access to the *exponential function*. The 'e' that appears here is the irrational number, $e \approx 2.72$. Thus, the exponential function takes an input x, and gives the output e^x. That is, it raises the number e to the input power. The graph of the exponential function is shown below.



Here are some important properties of the exponential function:

- The domain is $\mathbb{R}$.
- The range is $(0, \infty)$. In particular, e^x is *never* equal to zero.
- As x approaches infinity, so does e^x. (Which gets bigger *faster*, x or e^x?)
- As x approaches negative infinity, e^x approaches zero.

★★ In order to discuss the exponential function precisely, one must make sense of things like e^π; what is meant by e to an irrational power? There are a variety of approaches to resolving problems such as this. One approach is to show that the power series
$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
converges for all x, and then use this series to define e^x. Since multiplication and addition is defined for all real numbers, things like e^π then make sense.

Why the name 'exponential function'?

The *exponential function* is named as such, because it satisfies the familiar laws of exponents. For all real numbers x and y:

$$\begin{aligned}
e^0 &= 1 \\
e^x e^y &= e^{x+y} && \text{(same base, multiplied, add the exponents)} \\
\frac{e^x}{e^y} &= e^{x-y} && \text{(same base, divided, subtract the exponents)} \\
(e^x)^y &= e^{xy} && \text{(power to a power, multiply the exponents)} \\
e^{-x} &= \frac{1}{e^x}
\end{aligned}$$

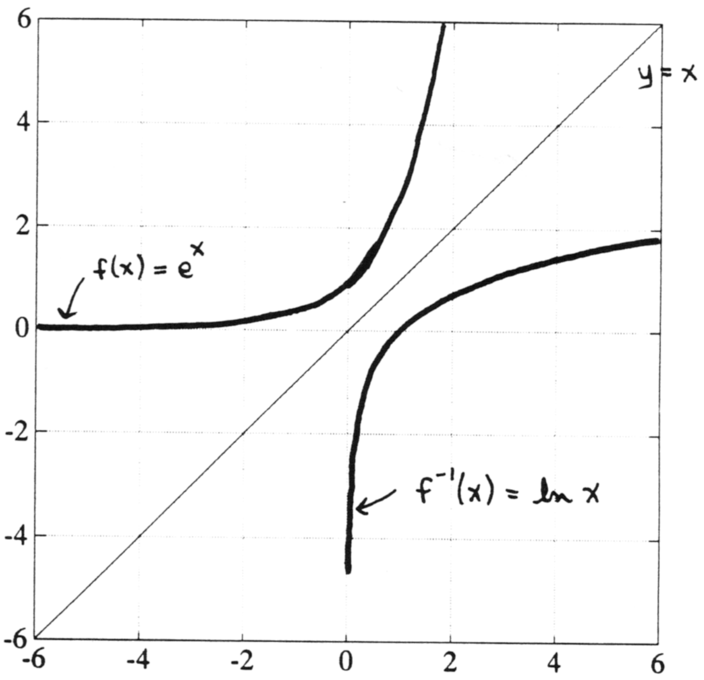
the natural logarithm function;

$\ln x$

Give the exponential function the name f; that is, define $f(x) := e^x$. Observe that f is 1-1. Therefore, it has an inverse f^{-1}. The name given to this inverse is *the natural logarithm function*, with output denoted by $\ln x$. That is, $f^{-1}(x) = \ln x$.

The *name* of the natural logarithm function is 'ln'; when the input is x, the output is often denoted by '$\ln x$' instead of '$\ln(x)$'. That is, the parentheses that are usually used in function notation are suppressed. However, in cases that might be ambiguous, parentheses *are* used: for example, '$\ln 2x + 5$' is ambiguous; does the author want '$(\ln 2x) + 5$', or '$\ln(2x + 5)$' ? In cases like this, parentheses are used to avoid confusion.

The key on your calculator labeled 'ln' (or, on some calculators, 'log') gives you access to the natural logarithm function. (Check that you've got the right key by finding $\ln 2.72$. You should get a number very close to 1, NOT close to 0.4!) The graph can be formed by reflecting the graph of $f(x) = e^x$ about the line $y = x$; this graph is shown below.



EXERCISE 6

Use your knowledge that the natural logarithm function is the inverse of the exponential function to answer the following questions:

♣ 1. What is the domain of the natural logarithm function?

♣ 2. What is the range of the natural logarithm function?

♣ 3. As x approaches infinity, what happens to $\ln x$?

♣ 4. As x gets closer and closer to zero (from the right), what happens to $\ln x$?

the classic 'inverse pair'

Since $f(x) = e^x$ and $f^{-1}(x) = \ln x$ are inverse functions, they 'undo' each other. That is:

$$f^{-1}(f(x)) = x \ \ \forall\, x \in \mathcal{D}(f)$$

Rewriting in terms of e^x and $\ln x$, we have:

$$\ln(e^x) = x \ \ \forall\, x \in \mathbb{R}$$

Also:

$$f(f^{-1}(x)) = x \ \ \forall\, x \in \mathcal{R}(f)$$

That is:

$$e^{\ln x} = x \ \ \forall\, x \in (0, \infty)$$

EXERCISE 7

Let $f(x) = e^x$, so that $f^{-1}(x) = \ln x$.

♣ 1. Check that the correct translation of

$$f^{-1}(f(x)) = x \ \ \forall\, x \in \mathcal{D}(f)$$

is:

$$\ln(e^x) = x \ \ \forall\, x \in \mathbb{R}$$

♣ 2. Check that the correct translation of

$$f(f^{-1}(x)) = x \ \ \forall\, x \in \mathcal{R}(f)$$

is:

$$e^{\ln x} = x \ \ \forall\, x \in (0, \infty)$$

uses for the exponential and logarithmic functions

The exponential and natural logarithm functions arise in many practical applications in business and the life sciences. An application of the exponential function is addressed in the next example.

simple interest

Suppose that \$2,000 is put in a bank offering an annual interest rate of 10%. After one year, interest of (0.10)(\$2000) = \$200 is earned, so the total amount in the bank is:

$$\$2000 + \$200 = \$2200$$

Remember that simple interest is computed using the formula:

$$\text{INTEREST} = (\text{PRINCIPAL})(\text{RATE})(\text{TIME})$$

The units must agree; if the principal is in dollars, and the rate is annual (say, $\frac{10\%}{\text{year}}$), then time must have units of years. In this case, the time units will cancel, leaving units of:

$$(\$)(\frac{1}{\text{year}})(\text{year}) = \$$$

Simple annual interest is not very desirable: who wants to wait an entire year before having any interest added in?

compounding

Most banks offer 'compounding'; that is, they add in the accumulated interest at regular intervals.

For example, suppose a bank offers 10% annual interest, compounded semi-annually. Then, after 6 months (0.5 years), the accumulated interest of

$$(\text{principal})(\text{rate})(\text{time}) = (2000)(.10)(.5) = 100$$

will be added in, yielding a total of:

$$2000 + (2000)(.10)(.5) = 2100$$

It is conventional to suppress all units for intermediate calculations.

After six more months, interest of $(2100)(.10)(.5) = 105$ will be added in, yielding a total of:

$$2100 + 105 = 2205$$

This is \$5 more than the amount obtained with simple annual interest.

EXERCISE 8

♣ Figure out how much will be in the bank after one year, if compounded quarterly (every 3 months).

continuous compounding

The best situation that can occur is if, *at each instant*, the accumulated interest is added in. This is called *continuous compounding*. It can be shown that the exponential function describes this situation! That is, P dollars, compounded continuously at an annual interest rate r, for t years, will grow to Pe^{rt} dollars. For example, \$2000 at 10% annual interest rate, compounded continuously, will grow to $2000e^{(.10)(1)} = \$2,210.34$ after one year!

EXERCISE 9

Suppose \$5,000 is put in a bank. How much will there be after 2 years, assuming:

♣ 1. 8% simple annual interest?

♣ 2. 8% annual interest, compounded semi-annually?

♣ 3. 8% annual interest, compounded continuously?

♣ 4. How much money is gained by having continuous compounding, as opposed to simple annual interest?

we will see the exponential and natural logarithm functions again

There are further properties of the exponential and natural logarithm functions that make them useful tools in calculus. Thus, we will see these functions again as we proceed throughout this text.

On to calculus!

These first two chapters, now drawing to a close, have been preparatory chapters. Some basic algebra skills were reviewed. Elements of the language of mathematics that will be needed throughout the course were developed. Finally, we are ready to begin the study of calculus.

The central idea in calculus is that of a *limit*. Without this concept, it would be impossible to speak precisely about continuity, differentiation, or integration. Thus, the next chapter begins with the study of *limits*.

QUICK QUIZ
sample questions

1. Is $f(x) = x^2$ a one-to-one function? Why or why not?
2. Translate this mathematical sentence:

$$\forall\, y \in \mathcal{R}(f), \quad \exists\,!\, x \in \mathcal{D}(f)$$

   What condition is being described here?
3. Sketch the graph of a function f satisfying the following properties: f is one-to-one, $\mathcal{D}(f) = [0, \infty)$, $\mathcal{R}(f) = (-\infty, 0]$.
4. Write down the two equations that describe the relationship between a 1-1 function f and its inverse f^{-1}.
5. Show that $f(x) = \frac{1}{3}x - 1$ is 1-1 (say, by graphing). Then, find f^{-1}, using any appropriate method.

KEYWORDS
for this section

One-to-one function and defining condition, the symbols $\forall$, $\exists$, $!$, *inverse function* f^{-1}, *precise conditions describing the relationship between a function and its inverse, finding a formula for* f^{-1} *(in two ways), symmetry about the line* $y = x$, *graphing* f^{-1}, *the exponential function and its graph, the exponent laws, the natural logarithm function and its graph, conditions relating* e^x *and* $\ln x$.

END-OF-SECTION EXERCISES

♣ Classify each entry below as an expression (EXP) or a sentence (SEN). The context will determine the appropriate variable type (number, set, function).

♣ For any *sentence*, state whether it is TRUE (T), FALSE (F), or CONDITIONAL (C).

1. $f^{-1}(x)$
2. $\forall\, y \in \mathcal{R}(f), \quad \exists\,!\, x \in \mathcal{D}(f)$
3. If f and f^{-1} are inverse functions, then $f(f^{-1}(x)) = x \ \ \forall\, x \in \mathcal{R}(f)$.
4. If f and f^{-1} are inverse functions, then $f(f^{-1}(y)) = y \ \ \forall\, y \in \mathcal{R}(f)$.
5. $f(f^{-1}(x))$
6. f is 1-1
7. For all real numbers x, $\ln(e^x) = x$.
8. For all real numbers x, $e^{\ln x} = x$.
9. $\ln x$
10. $e^x = 3$

Sketch the graphs of the following functions, by 'building them up' from simpler pieces. Find the domain and range of each function.

11. $f(x) = 3x^2 - 2$
12. $g(x) = \ln(x + 2)$
13. $h(x) = e^{x+3} + 5$
14. $f(x) = -\ln(x - 4)$