3.2 Limits—Making It Precise

a more precise way to view the sentence $\lim\limits_{x \to c} f(x) = l$

In the previous section, we said that when the sentence $\lim_{x \to c} f(x) = l$ is true, this means, roughly, that when x is close to c, then $f(x)$ is close to l.

Here's a more precise way to view this limit:

When the sentence $\lim_{x \to c} f(x) = l$ is true, then we can get the function values $f(x)$ *as close to* l *as desired*, merely by requiring that x be sufficiently close to c.

This idea is explored further in the next 'challenge'.

a friendly challenge

Suppose you are having a 'friendly' argument with a classmate. You have stated:

$$\lim_{x \to 2} 3x = 6$$

Your friend says: Prove it to me! Here's the resulting conversation:

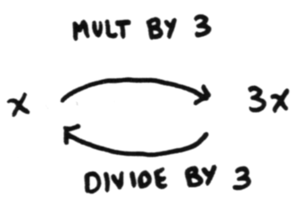
You: Okay, I will. I claim that I can get $3x$ *as close to* 6 as you want, just by requiring that x be close enough to 2. How close would you like to get $3x$ to 6?

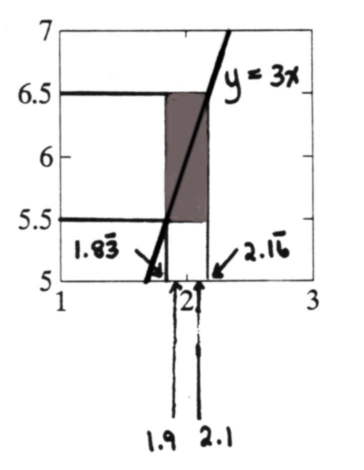
Friend: Within 0.5.

You: No problem. (Thinking out loud. . .) For the function $f(x) = 3x$, when x is the input, $3x$ is the output. So, to go from input to output, we multiply by 3. To go from output to input, divide by 3.

Now, you want me to get the output $3x$ within 0.5 of 6. That is, you want $3x$ to be in the interval $(6-0.5, 6+0.5) = (5.5, 6.5)$. The output 5.5 corresponds to the input $\frac{5.5}{3} = 1.8\overline{3}$; the output 6.5 corresponds to the input $\frac{6.5}{3} = 2.1\overline{6}$. As long as I keep x within the interval $(1.8\overline{3}, 2.1\overline{6})$, then $3x$ will be within the requested interval. We don't even need to 'cut things so close'. As long as x is within, say, 0.1 of 2, then $3x$ will be well within 0.5 of 6.

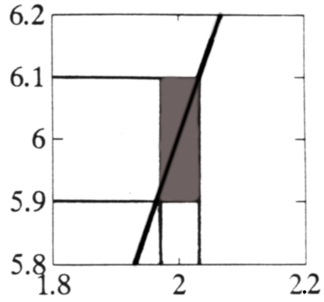
(Look at the sketch below.)





Friend: Well, now I want $3x$ no further than 0.1 from 6.

You: So, you want $3x$ to be in the interval $(5.9, 6.1)$. Well, $\frac{5.9}{3} \approx 1.97$, and $\frac{6.1}{3} \approx 2.03$. To be safe, let's just keep x within the interval $(1.98, 2.02)$. That is, as long as x is within 0.02 of 2, then $3x$ will be well within 0.1 of 6.

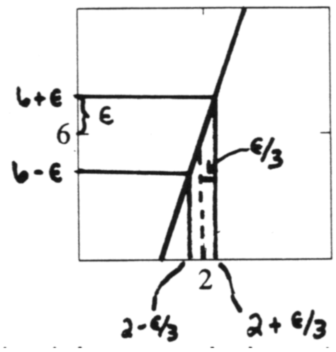


Friend: Well—now I want $3x$ within 0.0001 of 6!

You: (Calculates.) Just keep x within, say, 0.00003 of 2.

Friend: I'm thinking of a *really small number*; call it ϵ. I want $f(x)$ within ϵ of l.

You: (Draws the sketch below for Friend.) Just keep x within $\frac{\epsilon}{3}$ of 2.



Keep this 'challenge' in mind as you study the precise definition of the limit statement.

DEFINITION

the limit of a function

$\lim_{x\to c} f(x) = l$

The sentence '$\lim_{x\to c} f(x) = l$' is defined by:

$$\lim_{x\to c} f(x) = l \iff \text{For every } \epsilon > 0, \text{ there exists } \delta > 0, \text{ such that if } 0 < |x - c| < \delta \text{ and } x \in \mathcal{D}(f), \text{ then } |f(x) - l| < \epsilon.$$

In this definition, two sentences are being compared with the 'is equivalent to' symbol, $\iff$. The sentence on the left is being given meaning by the sentence on the right. These two sentences always have precisely the same truth values, and hence can be used interchangeably. More precisely, the sentence

$$\lim_{x\to c} f(x) = l \tag{*}$$

on the left is being *defined* by the sentence

$$\begin{array}{l}\text{For every } \epsilon > 0, \text{ there exists } \delta > 0, \text{ such that if}\\ 0 < |x-c| < \delta \text{ and } x \in \mathcal{D}(f), \text{ then } |f(x) - l| < \epsilon.\end{array} \tag{**}$$

on the right. This is how we determine the truth value of the sentence (*). If (**) is true, then so is (*). If (**) is false, then so is (*).

Next, we must carefully investigate (**), to see when it is true.

How close do you want $f(x)$ to be to l?

(within ϵ)

The sentence (**) begins with:

$$\text{For every } \epsilon > 0 \quad \cdots$$

Think of ϵ as being a small positive number, that says *how close you want the function values $f(x)$ to be to l*. In order for the sentence (**) to be true, the remainder of this sentence is going to have to be true for *every positive number* ϵ. In particular, it's going to have to be true when ϵ is arbitrarily close to zero; like $\epsilon = 10^{-2000}$.

The remainder of the sentence (**) addresses the question: How close do we need to get x to c in order to get $f(x)$ within ϵ of l?

How close must x be to c to accomplish this?

(within δ)

The sentence continues:

$$\cdots \quad \text{there exists } \delta > 0 \quad \cdots$$

Think of δ as a small positive number that says *how close we must get the x values to c, to ensure that the corresponding outputs fall within ϵ of l*. For most functions, the smaller ϵ is, the smaller δ is going to have to be.

check that δ really works

The sentence continues:

$$\cdots \quad \text{such that if } 0 < |x-c| < \delta \text{ and } x \in \mathcal{D}(f) \quad \cdots$$

Up to this point, we have a 'challenge'; we want to get the function values within ϵ of l. We also have a 'proposed solution'; just keep the x values within δ of c. Now, we're going to show that this δ really works.

So we're saying: suppose x is a number that makes '$0 < |x-c| < \delta$ and $x \in \mathcal{D}(f)$' true. What x will make both of these sentences true? Well, $0 < |x-c| < \delta$ says that x must lie within δ of c, but not equal c. Remember that we don't ever let x equal c when evaluating a limit—x just gets arbitrarily close to c. Also, in order to talk about $f(x)$, we must certainly have x in the domain of f. Hence the sentence $x \in \mathcal{D}(f)$.

The sentence finishes:

$$\cdots \quad \text{then } |f(x) - l| < \epsilon.$$

As long as x is in the domain of f, and sufficiently close to c (but not equal to c), then the sentence $|f(x) - l| < \epsilon$ will be true. When is $|f(x) - l| < \epsilon$ true? Exactly when the distance between $f(x)$ and l is less than ϵ.

Rephrasing, in English

The mathematical sentence

$$\text{For every } \epsilon > 0, \text{ there exists } \delta > 0, \text{ such that if } 0 < |x - c| < \delta \text{ and } x \in \mathcal{D}(f), \text{ then } |f(x) - l| < \epsilon. \tag{**}$$

can be stated in English, as follows:

For every number epsilon greater than zero, there exists a number delta greater than zero, with the property that if x is within delta of c, but not equal to c, and if x is in the domain of f, then the distance between $f(x)$ and l is less than epsilon.

You must know the precise definition of the sentence:

$$\lim_{x \to c} f(x) = l$$

Also, you must be able to explain, in English, what this sentence means.

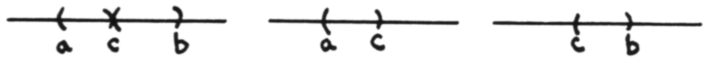
★ *f must be defined on some interval near c*

To avoid complications, we will only consider limits $\lim_{x \to c} f(x)$ in situations where f is defined on some interval near c; this interval may or may not include c.

More precisely, f must be defined at least on an interval of the form:

$$(a, c) \cup (c, b) \quad \text{or} \quad (a, c) \quad \text{or} \quad (c, b)$$

where $a < c$ and $c < b$.



The numbers a and b may, however, be arbitrarily close to c.

limits are fundamental to calculus

The concept of the limit of a function is fundamental to calculus. You will see limits again when we talk about continuous functions; when we talk about differentiating; when we talk about integrating. To truly *understand* calculus, you must *understand* limits.

The following examples and exercises should help in the learning process.

EXAMPLE

using the precise definition of the limit of a function

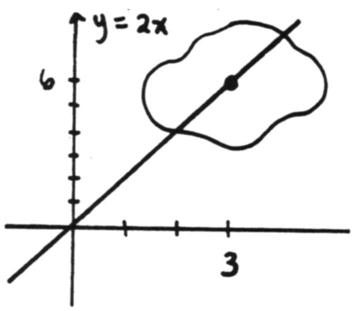
Problem: Use the precise definition of the limit of a function, to argue that:

$$\lim_{x \to 3} 2x = 6$$

Solution:

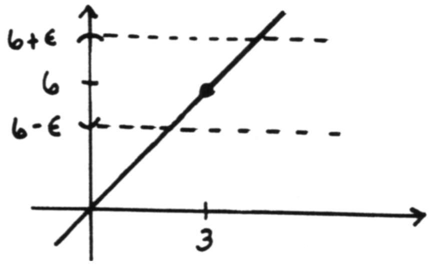
Step 1; state what you need to show, and sketch the function near c

Step 1. (State what you need to show, and sketch the function near c.) It must be shown that we can get $2x$ as close to 6 as desired (within ϵ), by requiring that x be sufficiently close to 3 (within δ).



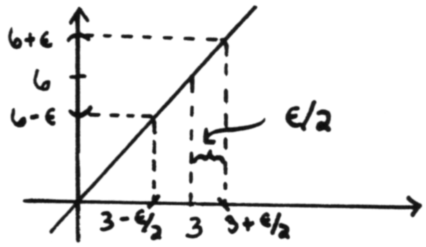
Step 2; set up an ϵ-interval about l

Step 2. (Set up an ϵ-interval about l.) Let $\epsilon > 0$. We want to get $2x$ within ϵ of 6; that is, within the interval $(6-\epsilon, 6+\epsilon)$. Show this interval on your sketch.



Step 3; 'pull back' to an appropriate interval about c

Step 3: ('Pull back' to an appropriate interval about c.) With the function $f(x) = 2x$, to go from an input x to an output $2x$, we multiply by 2; and to go from an output $2x$ to an input x, we divide by 2.

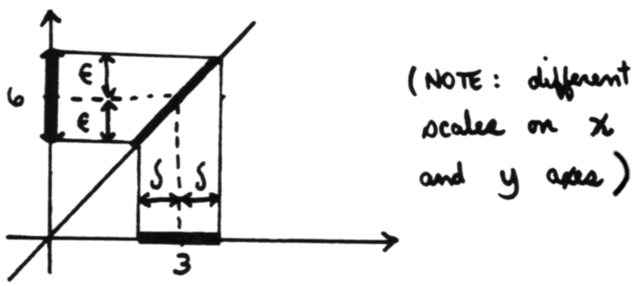


So, when the output is $6+\epsilon$, the corresponding input is $\frac{6+\epsilon}{2} = 3 + \frac{\epsilon}{2}$. And, when the output is $6-\epsilon$, the input is $\frac{6-\epsilon}{2} = 3 - \frac{\epsilon}{2}$.

Alternately, solving the equation $y = 2x$ for x yields $x = y/2$. When $y = 6+\epsilon$, we have $x = \frac{6+\epsilon}{2} = 3 + \frac{\epsilon}{2}$; and when $y = 6-\epsilon$, we have $x = \frac{6-\epsilon}{2} = 3 - \frac{\epsilon}{2}$.

Step 4; summarize results, by stating the 'δ that works'

Step 4. (Summarize your results, by stating the 'δ that works'.) Thus, if we take δ to be $\epsilon/2$, then whenever x is within δ of 3, we will have $2x$ within ϵ of 6.



the 4-step process for investigating limits

You must be able to investigate limits in the manner discussed in the previous example. Always follow the basic 4-step process that leads to a 'δ that works', and always make a sketch that summarizes what you are doing.

EXERCISE 1

elaborating on the previous example

♣ 1. In the preceding example, could δ have been taken to be a positive number *less than* $\frac{\epsilon}{2}$? Why or why not?

♣ 2. Is $\frac{\epsilon}{2} - .01$ necessarily a positive number less than $\frac{\epsilon}{2}$? Why or why not?

♣ 3. Is $\frac{\epsilon}{3}$ necessarily a positive number less than $\frac{\epsilon}{2}$? Why or why not?

♣ 4. Write down two positive numbers that are (always) less than $\frac{\epsilon}{2}$.

EXERCISE 2

using the 4-step process to investigate a limit

♣ 1. Use the 4-step process to show that the following limit statement is true:

$$\lim_{x\to 2} 4x = 8$$

DO NOT just 'copy' the preceding example. Close your book, and try to write down the argument yourself. If you get stuck, then re-read the previous example, and see where you went wrong. But then close your book again.

♣ 2. Use the 4-step process to show that the following limit statement is true:

$$\lim_{x\to 1} 2x + 3 = 5$$

EXERCISE 3

the limit of a constant function

Let f be the constant function defined by $f(x) = 5$.

♣ 1. Describe, in words, what the function f does.

♣ 2. Draw a 'black box' that describes f.

♣ 3. Does $\lim_{x\to 2} f(x)$ exist? Why or why not?

♣ 4. Use the 4-step process to show that the following limit statement is true:

$$\lim_{x\to 2} f(x) = 5$$

♣ 5. What did you choose for δ? Are there other 'natural' choices for δ?

EXAMPLE

evaluating a more general limit

Problem: Use the 4-step process to show that the following limit statement is true:

$$\lim_{x\to 2} x^3 = 8$$

Step 1

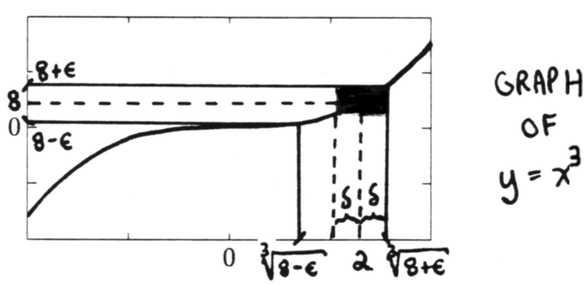
Step 1. It must be shown that we can get x^3 as close to 8 as desired (within ϵ), by requiring that x be sufficiently close to 2 (within δ).

Step 2

Step 2. Let $\epsilon > 0$. We want to get x^3 within ϵ of 8; that is, in the interval $(8-\epsilon, 8+\epsilon)$.

Step 3

Step 3. Refer to the 'mapping diagram' below. When the output is $8+\epsilon$, the corresponding input is $\sqrt[3]{8+\epsilon}$; when the output is $8-\epsilon$, the input is $\sqrt[3]{8-\epsilon}$.



Step 4

Step 4. Now, which is the shorter distance: from 2 to $\sqrt[3]{8+\epsilon}$ or from 2 to $\sqrt[3]{8-\epsilon}$? Since the curve $y = x^3$ rises more steeply as x gets larger, the shorter distance is from 2 to $\sqrt[3]{8+\epsilon}$. (This fact will be proven later on in the course.) Thus, take the shorter distance $(\sqrt[3]{8+\epsilon} - 2)$ to be δ.

Then, as long as x is within δ of 2, we will have x^3 within ϵ of 8.

EXERCISE 4

♣ 1. Use the 4-step process to show that the following limit statement is true:

$$\lim_{x \to 3} x^3 = 27$$

♣ 2. Use the 4-step process to show that the following limit statement is true:

$$\lim_{x \to c} x^3 = c^3$$

where c is any positive real number.

EXERCISE 5

♣ 1. Let $c < 0$. Use the 4-step process to show that the following limit statement is true:

$$\lim_{x \to c} x^3 = c^3$$

♣ 2. Use the 4-step process to show that the following limit statement is true:

$$\lim_{x \to 2} x^2 = 4$$

One of the beautiful things about the precise definition, is that now we don't have to worry about whether or not the function is defined on 'both sides' of c; the definition takes care of this for us, by requiring that x must be in the *domain* of the function! This is illustrated in the next example.

EXAMPLE

investigating a limit, when f is only defined on one side of c

Problem: Use the 4-step process to show that the following limit statement is true:

$$\lim_{x \to 0} \sqrt{x} + 3 = 3$$

Step 1

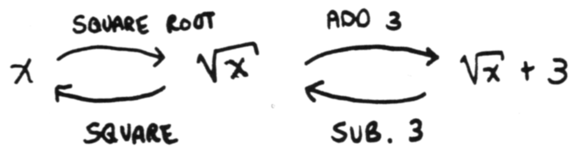
Step 1. Define $f(x) := \sqrt{x} + 3$. It must be shown that we can get $\sqrt{x} + 3$ as close to 3 as desired (within ϵ), merely by requiring that x be *in the domain of f*, and sufficiently close to 0 (within δ).

Step 2

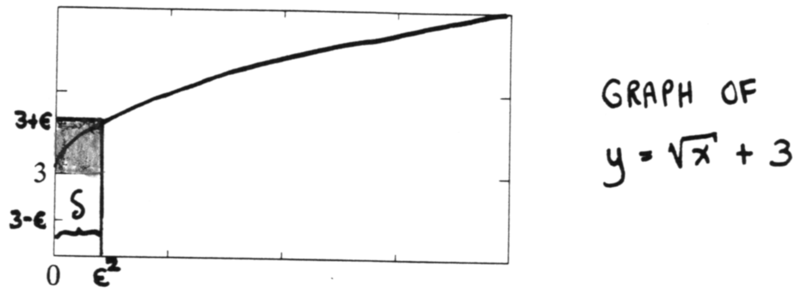
Step 2. Let $\epsilon > 0$. We must get $\sqrt{x} + 3$ within ϵ of 3; that is, within the interval $(3 - \epsilon, 3 + \epsilon)$.

Step 3

Step 3. Refer to the 'mapping diagram' below:



When the output is $3+\epsilon$, the corresponding input is $((3+\epsilon)-3)^2 = \epsilon^2$. There is no input corresponding to the output $3-\epsilon$.



Step 4

Step 4. Referring to the sketch, we see that whenever x is within ϵ^2 of 0, and is within the domain of f, then $f(x)$ will be within ϵ of 3.

So, take $\delta = \epsilon^2$.

EXERCISE 6

♣ 1. Use the 4-step process to show that the following limit statement is true:

$$\lim_{x\to 0} \sqrt{x} + 2 = 2$$

♣ 2. Use the 4-step process to show that the following limit statement is true:

$$\lim_{x\to 2} f(x) = 4$$

where $f\colon [2,\infty) \to \mathbb{R}$ is defined by $f(x) = x^2$.

EXAMPLE

investigating a limit, when f is defined 'in a strange way' at c

The precise definition also 'covers' the situation when the function is defined in a 'strange way' at c. For example, consider the function f given by:

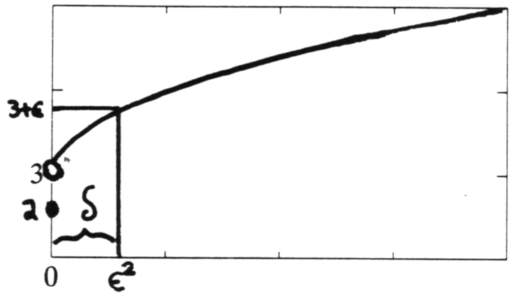
$$f(x) = \begin{cases} \sqrt{x} + 3 & x > 0 \\ 2 & x = 0 \end{cases}$$

Again, $\lim_{x\to 0} f(x) = 3$. We already found the 'δ that works' in the previous example. Then, as long as:

- x is within δ of 0
- x is in the domain of f
- x is *not equal to* 0

then $f(x)$ will be within ϵ of 3.

In mathematical language: if $0 < |x| < \delta$ and $x \in \mathcal{D}(f)$, then $|f(x) - 3| < \epsilon$.



GRAPH OF f

EXERCISE 7

♣ 1. Which part of the sentence

$$0 < |x| < \delta$$

says that x must be within δ of 0?

♣ 2. Which part of the sentence says that x must not equal 0?

EXERCISE 8

Let $f\colon [1,\infty) \to \mathbb{R}$ be defined by the rule:

$$f(x) = \begin{cases} 3x-5 & \text{for } x > 1 \\ 1 & \text{for } x = 1 \end{cases}$$

♣ 1. Graph f.

♣ 2. What is the domain of f? In particular, is f defined at $x = 1$?

♣ 3. Use the 4-step process to show that the following limit statement is true:

$$\lim_{x\to 1} f(x) = -2$$

♣ 4. Discuss the phrase

$$0 < |x-c| < \delta \quad \text{and} \quad x \in \mathcal{D}(f)$$

relative to this example. What is c? What is δ? For what values of x is this phrase true, for the function f being considered here?

one-sided limits

Sometimes one is only interested in investigating the function values $f(x)$ as x approaches c from *only one side* (right or left), even though f *may* be defined on both sides of c. In other words, one can ask the question: as x approaches c from one side (right or left), do the corresponding function values $f(x)$ approach any particular real number? This leads to the notion of *one-sided limits.* Here's the precise definition of the *right-hand limit*:

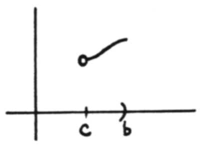
DEFINITION
right-hand limit

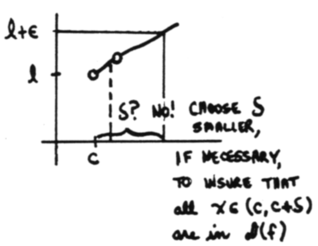
Let f be a function that is defined at least on an interval of the form (c, b), where $b > c$. Then:

$$\lim_{x\to c^+} f(x) = l \iff \text{For every } \epsilon > 0, \text{ there exists } \delta > 0, \text{ such that if } x \in (c, c+\delta), \text{ then } |f(x) - l| < \epsilon.$$

The phrase '$x \to c^+$' is read as '*x approaches c from the right-hand side*' or '*x approaches c from the positive side*'.

investigating this definition





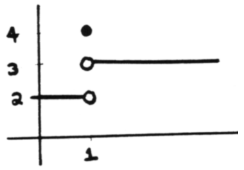
Let's investigate this definition. Here are the ways that it differs from the earlier (two-sided) limit:

- Since it is desired to let x approach c from the right-side, we require that f be defined at least on some small interval to the right of c.

- For the limit to exist (and equal l), we must be able to get $f(x)$ as close to l as desired (within ϵ), by requiring that x be close enough to c (within δ), *on the right-hand side.* Note that whenever $x \in (c, c+\delta)$, then x lies to the right of c. Also, note that x is not allowed to equal c, since c is not included in the interval $(c, c+\delta)$.

- The phrase '$x \in \mathcal{D}(f)$' was needed in the definition of the two-sided limit to 'cover the cases' when f was not defined on both sides of c. Now, however, we are *assuming* that f is defined to the right of c, so the phrase is not necessary. Delta (δ) can always be chosen small enough so that x will lie in the domain of f.

EXAMPLE
investigating a right-hand limit



Consider the function f given by:

$$f(x) = \begin{cases} 2 & x < 1 \\ 3 & x > 1 \\ 4 & x = 1 \end{cases}$$

The graph of f is shown. In this case, the (two-sided) limit $\lim_{x \to c} f(x)$ does not exist. (Why?) However:

$$\lim_{x \to 1^+} f(x) = 3$$

EXERCISE 9
left-hand limit

♣ 1. After studying the definition of right-hand limit, write down a precise definition of a *left-hand limit*. Denote the left-hand limit by

$$\lim_{x \to c^-} f(x)$$

and read the phrase '$x \to c^-$' as x *approaches* c *from the left-hand side* or x *approaches* c *from the negative side*. Be sure to write complete mathematical sentences.

Now, consider the function f given by:

$$f(x) = \begin{cases} x^2 & x < 3 \\ 5 & x > 3 \\ 7 & x = 3 \end{cases}$$

♣ 2. Graph this function f.

♣ 3. Evaluate the limits:

$$\lim_{x \to 3^+} f(x) \quad \text{and} \quad \lim_{x \to 3^-} f(x)$$

♣ 4. Why doesn't the (two-sided) limit $\lim_{x \to 3} f(x)$ exist?

♣ 5. Redefine the function for $x > 3$ so that the two-sided limit does exist for the redefined function.

In the next section, some tools will be developed to help us work with limit statements. Then, we will be in a position to discuss *continuity* precisely.

QUICK QUIZ

sample questions

1. Give a precise definition of the limit statement $\lim_{x\to c} f(x) = l$.
2. Use the 4-step process to show that the following limit statement is true:

$$\lim_{x\to -1} 3x = -3$$

3. Let f be defined by:

$$f(x) = \begin{cases} x^2 & \text{for } \ x > 2 \\ 6 & \text{for } \ x = 2 \\ x & \text{for } \ x < 2 \end{cases}$$

Sketch the graph of f, and evaluate the following limits, if they exist:

$$\lim_{x\to 2} f(x)\ , \qquad \lim_{x\to 2^+} f(x)\ , \qquad \lim_{x\to 2^-} f(x)$$

KEYWORDS

for this section

The precise definition of:

$$\lim_{x\to c} f(x) = l$$

You must be able to explain this definition in words, and with appropriate sketches. You must be able to use the 4-step process to show that certain limit statements are true. Also, you must understand one-sided limits.

END-OF-SECTION EXERCISES

Use the 4-step process to show that the following limit statements are true:

1. $\displaystyle\lim_{x\to -2}(-x^2) = -4$

2. $\displaystyle\lim_{x\to 2}(2 - x^3) = -6$

3. $\displaystyle\lim_{t\to 16}\sqrt[4]{t} = 2$

4. $\displaystyle\lim_{t\to 0}|t| = 0$

Graph the given function. Then, evaluate the specified limits. If a limit does not exist or is not defined, so state.

5. $f\colon (1,\infty)\to\mathbb{R},\ \ f(x) = x^2+2;\quad \displaystyle\lim_{x\to 1} f(x),\ \ \lim_{x\to 1^+} f(x),\ \ \lim_{x\to 1^-} f(x)$

6. $f\colon (-\infty,1)\to\mathbb{R},\ \ f(x) = x^2+2;\quad \displaystyle\lim_{x\to 1} f(x),\ \ \lim_{x\to 1^+} f(x),\ \ \lim_{x\to 1^-} f(x)$

7. $$g(x) = \begin{cases} x & \text{for } \ x < -1 \\ 2 & \text{for } \ x = -1 \\ -x^2 & \text{for } \ x > -1\ ; \end{cases}$$

$\displaystyle\lim_{x\to -1} g(x),\ \ \lim_{x\to -1^+} g(x),\ \ \lim_{x\to -1^-} g(x)$

8. $$g(x) = \begin{cases} 3 & \text{for } \ x \ge 2 \\ 1 & \text{for } \ x < 2\ ; \end{cases}$$

$\displaystyle\lim_{x\to 2} g(x);\ \ \lim_{x\to 2^+} g(x);\ \ \lim_{x\to 2^-} g(x)$

9. True or False: if $\displaystyle\lim_{x\to c} f(x)$ exists and f is defined on both sides of c, then both $\displaystyle\lim_{x\to c^+} f(x)$ and $\displaystyle\lim_{x\to c^-} f(x)$ exist.

10. True or False: if both one-sided limits $\displaystyle\lim_{x\to c^+} f(x)$ and $\displaystyle\lim_{x\to c^-} f(x)$ exist, then $\displaystyle\lim_{x\to c} f(x)$ exists.