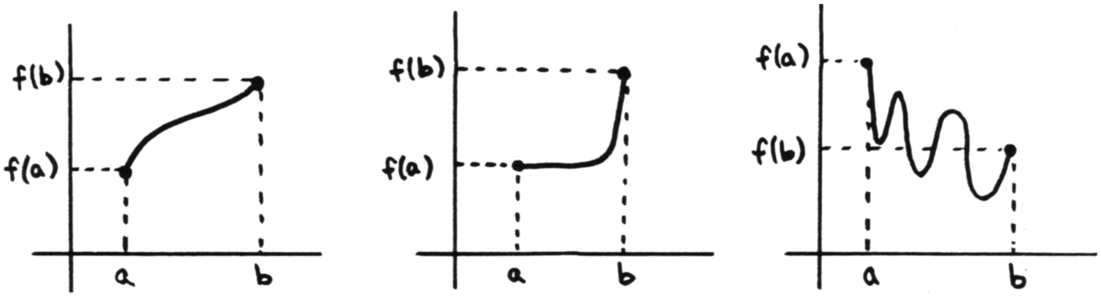
3.6 The Intermediate Value Theorem

Introduction

This section and the next present two fundamental properties of functions that are continuous on a closed interval.

the Intermediate Value Theorem

In this section, the *Intermediate Value Theorem* is discussed. Roughly, it says that a function continuous on $[a, b]$ must take on all values between $f(a)$ and $f(b)$; that is, all *intermediate* values. The idea is simple: since f is continuous on $[a, b]$, whenever the inputs are close, so must be the outputs. So if one begins at the point $(a, f(a))$ and traces the function, it is impossible to reach the point $(b, f(b))$ without passing through all y-values between $f(a)$ and $f(b)$. This idea is illustrated below.



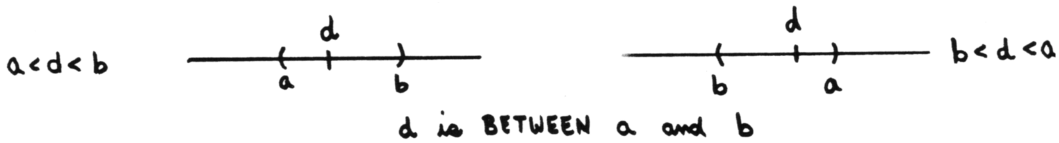
the word 'between'; d is between a and b

Precise statements of the Intermediate Value Theorem usually use the word 'between'. Here's the mathematical meaning of the word 'between':

Given real numbers a and b, one says that d is *between* a *and* b if:

- $a \ne b$, and
- d lies in the open interval bounded by a and b

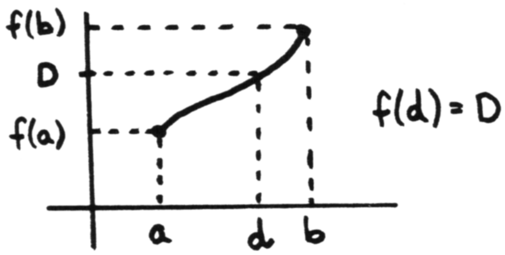
Perhaps a better phrase would be 'strictly between'; however, this is not in common usage.



THEOREM

the Intermediate Value Theorem (IVT)

Let f be continuous on $[a, b]$. If D is any number between $f(a)$ and $f(b)$, then there exists a number d between a and b with $f(d) = D$.



the IVT is an EXISTENCE theorem

The Intermediate Value Theorem is an *existence* theorem. That is, under appropriate hypotheses, it guarantees the *existence* of a number with a certain property.

Be a good reader!

check that all the hypotheses are really needed

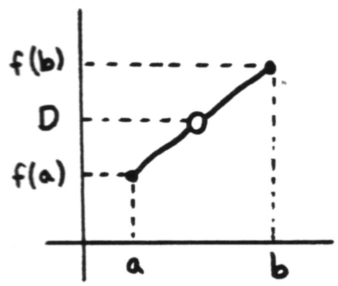
When presented with a theorem, a good reader will 'play with' the hypotheses, to see if they are all really needed to obtain the stated result.

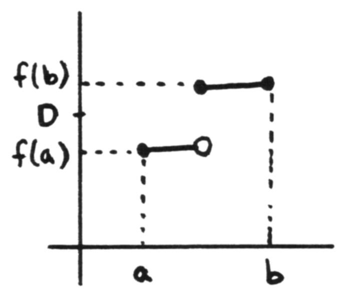
Remember that the *hypotheses* of a theorem are the things that are assumed to be true. The singular form of '*hypotheses*' is '*hypothesis*'. The *hypothesis* of the Intermediate Value Theorem is that f is continuous on $[a, b]$. Remember that this requires that f be defined on $[a, b]$, continuous on (a, b), and well-behaved at the endpoints. Are all these requirements really necessary?

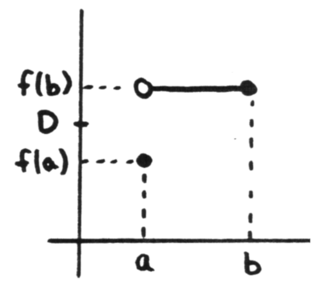
The sketches below illustrate that they are.

The first and second sketches illustrate situations where there is no d between a and b with $f(d) = D$. In both cases, f is not continuous at each point in the open interval (a, b).

The third sketch illustrates another situation where there is no d between a and b with $f(d) = D$. Here, f is not well-behaved at the left-hand endpoint, a.



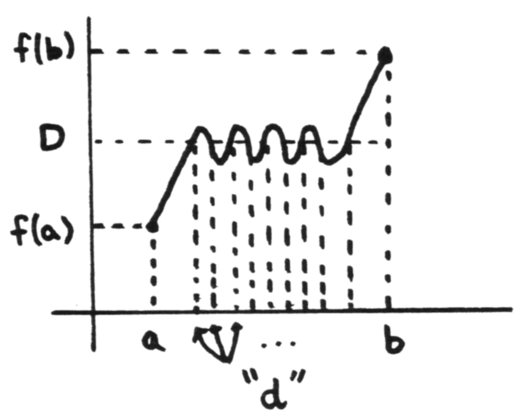




d may NOT be unique

Note that the Intermediate Value Theorem is NOT a uniqueness theorem. It only guarantees the *existence* of a certain number; it makes no claims about 'how many' such numbers there may be.

Indeed, the sketch below illustrates that an 'intermediate value' may be taken on ANY given number of times.



EXERCISE 1
practice with the IVT

♣ 1. Sketch the graph of a function f that satisfies the following requirements: f is continuous on $[a, b]$, $f(a) = 3$, $f(b) = 4$. Must there be a number $c \in (a, b)$ with $f(c) = \pi$? Why or why not? If so, label it on your graph.

♣ 2. Suppose f is continuous at each point in $(-5, 5)$. Must f be continuous on the interval $[-2, 2]$? Why or why not? Can you generalize this example?

♣ 3. Suppose f is continuous on $[a, b]$, $f(a) < 0$ and $f(b) > 0$. Must there exist a number c between a and b with $f(c) = 0$? Why or why not?

♣ 4. Suppose that $f(0) = 2$, $f(1) = 3$, but there is no number between 0 and 1 with function value 2.5. What conclusion, if any, can you make?

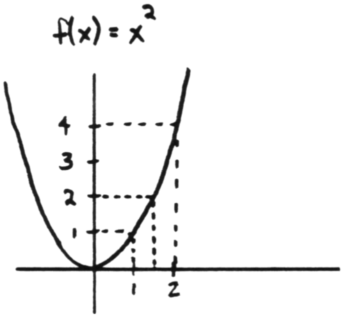
EXAMPLE
using the IVT to guarantee existence of a solution to an equation, and estimate its value

This example illustrates a very common use of the Intermediate Value Theorem: to guarantee existence of a solution to a given equation, and estimate the value of this solution.

Consider the equation $x^2 = 2$. We want a (real) number which, when squared, yields the number 2. How do we know that such a number exists? The Intermediate Value Theorem can be used to guarantee a solution, as follows:

Define $f(x) := x^2$. Then f is continuous on any interval $[a, b]$. We want to find a number x for which $f(x) = 2$. Since $f(1) = 1^2 = 1$ and $f(2) = 2^2 = 4$, there must exist $d_1 \in (1, 2)$ with $f(d_1) = 2$ (Why?) Thus, $d_1^2 = 2$, so d_1 is a solution of the equation $x^2 = 2$. Now we have existence of a solution to this equation that lies between 1 and 2; and it has been approximated within 1 unit. That is, the solution lies in an interval of length 1.

Knowing that a solution d_1 lies between 1 and 2 is okay, but it would be nice to get a better estimate of the number d_1. So, let's refine our approach. Let's make a table of some functions values:



a solution lies between 1.4 and 1.5

x	x^2
1.0000	1.0000
1.1000	1.2100
1.2000	1.4400
1.3000	1.6900
1.4000	1.9600
1.5000	2.2500
1.6000	2.5600
1.7000	2.8900
1.8000	3.2400
1.9000	3.6100
2.0000	4.0000

Since $f(1.4) = (1.4)^2 = 1.96$ and $f(1.5) = (1.5)^2 = 2.25$, the intermediate value theorem guarantees the existence of $d_2 \in (1.4, 1.5)$ with $d_2^2 = 2$. The solution has been approximated within 0.1.

One more time. Here's another table of function values:

x	x^2
1.4000	1.9600
1.4100	1.9881
1.4200	2.0164
1.4300	2.0449
1.4400	2.0736
1.4500	2.1025
1.4600	2.1316
1.4700	2.1609
1.4800	2.1904
1.4900	2.2201
1.5000	2.2500

a solution to $x^2 = 2$ lies between 1.41 and 1.42

Since $f(1.41) = 1.9881$ and $f(1.42) = 2.0164$, there must exist $d_3 \in (1.41, 1.42)$ with $d_3^2 = 2$. The solution has been approximated within 0.01. It is easy to see how this process can continue.

The exact solution to $x^2 = 2$ in the interval being investigated is, of course, $\sqrt{2} \approx 1.4142$.

EXERCISE 2

♣ 1. Use the Intermediate Value theorem to find a solution to the equation $x^4 - 8x^2 = -15$ that lies in the interval $[0, 2]$. Approximate the solution to within 0.01; that is, get an interval of length .01 that contains a solution.

♣ 2. Find another solution to $x^4 - 8x^2 = -15$ that lies in the interval $[2, 3]$. Approximate it to within 0.1.

♣ 3. Find the exact solutions to the equation $x^4 - 8x^2 = -15$. Be sure to write a complete mathematical sentence. Which of these solutions were you finding in parts (1) and (2)?

EXAMPLE

Now consider the equation $x^3 = 2x + 3$. Suppose it is desired to locate and estimate a solution of this equation. Since the variable x appears on both sides of the equation, the approach taken in the previous example must be modified. The notion of *equivalence* comes to the rescue.

Since

$$x^3 = 2x + 3 \quad \iff \quad x^3 - 2x - 3 = 0\ ,$$

these two equations have exactly the same truth values. They are interchangeable. We can work with whichever one is easier to work with. In this case, it is easier to work with $x^3 - 2x - 3 = 0$.

If we can find a value of x that makes $x^3 - 2x - 3 = 0$ true, then this same x will make $x^3 = 2x + 3$ true.

Define $f(x) := x^3 - 2x - 3$. A quick table shows that $f(1) = -4$ and $f(2) = 1$. Thus, the intermediate value theorem guarantees the existence of a number $d_1 \in (1, 2)$ with $f(d_1) = 0$. That is, $d_1^3 - 2d_1 - 3 = 0$.

x	f(x)
1	-4
2	1
3	18
4	53
5	112
6	201
7	326
8	493
9	708
10	977

1.0000	-4.0000
1.1000	-3.8690
1.2000	-3.6720
1.3000	-3.4030
1.4000	-3.0560
1.5000	-2.6250
1.6000	-2.1040
1.7000	-1.4870
1.8000	-0.7680
1.9000	0.0590
2.0000	1.0000

1.8000	-0.7680
1.8100	-0.6903
1.8200	-0.6114
1.8300	-0.5315
1.8400	-0.4505
1.8500	-0.3684
1.8600	-0.2851
1.8700	-0.2008
1.8800	-0.1153
1.8900	-0.0287
1.9000	0.0590

Another table of values shows that $f(1.8) = -0.7680$ and $f(1.9) = 0.0590$, so there must be a solution d_2 in $(1.8, 1.9)$.

Another table shows that $f(1.89) = -0.0287$ and $f(1.9) = 0.0590$. Thus, there must be a solution d_3 in $(1.89, 1.90)$.

Note that for this equation, it is not easy to find an exact solution.

EXERCISE 3

♣ Use the Intermediate Value Theorem to show the existence of a solution to the equation $x^3 - x^2 = 5x - 5$ that lies between 2 and 3. Then, approximate this solution to within 0.01.

EXERCISE 4

♣ 1. Suppose that f is continuous on $[a, b]$, and $f(a) = f(b) := D$. Must there be $d \in [a, b]$ with $f(d) = D$?

♣ 2. Suppose that f is continuous on $[a, b]$, and $f(a) = f(b) := D$. Must there be $d \in (a, b)$ with $f(d) = D$? Justify your answer. Be sure to write complete mathematical sentences.

EXERCISE 5

♣ 1. On a number line, show c, d and $\frac{c+d}{2}$ for various choices of c and d.

♣ 2. Let c and d be real numbers with $c < d$. Prove that $\frac{c+d}{2}$ is exactly half-way between c and d.

♣ 3. Suppose that f is continuous on $[a, b]$. Must there exist $d \in [a, b]$ with $f(d) = \frac{f(a)+f(b)}{2}$? Why or why not?

An 'implication' is a sentence of the form: If A, then B

The second sentence in the Intermediate Value Theorem is:

If D is any number between $f(a)$ and $f(b)$, then there exists a number d between a and b with $f(d) = D$.

This is a sentence of the form

If A, then B

IF $\boxed{D \text{ is any number between } f(a) \text{ and } f(b),}$

THEN $\boxed{\text{there exists a number } d \text{ between } a \text{ and } b \text{ with } f(d) = D.}$

A sentence of the form

If A, then B

is called an *implication.* Implications are the most common type of mathematical sentence. Therefore, to understand mathematics, you must understand implications. In this section, the study of implications begins. This study will continue throughout the text.

intuition for the sentence 'If A, then B'

Everyone is familiar with sentences of the form '*If A, then B*' because they are as common in English as they are in mathematics. Fortunately, there are a lot of similarities between the English and mathematical meanings. So let's review the English meaning, and then move on to the (more precise) mathematical meaning.

Suppose a person is trying to sell raffle tickets, and says to you,

If your ticket is chosen, then you'll get $1,000.

This is an (English) sentence of the form '*If A, then B*'. You know what it means: if the first part of the sentence is true—that is, if your ticket is chosen—then the second part of the sentence will also be true—you will get $1,000.

Suppose the big day arrives, and your ticket *isn't* selected. The first part of the sentence is *not true* in this case. And, you don't get $1,000. Does this make the person who sold you the raffle ticket into a liar? Of course not! The sentence

'If your ticket is chosen, then you'll get $1,000.'

is still true. It's just that this sentence only guarantees us that the second part will be true IF the first part is true!

another English example of an 'If A, then B' sentence

Here's another example. Suppose your parents say,

If you get an 'A' in calculus, then we'll take you out to dinner.

Now, if you get an 'A' and your parents *don't* take you out, then they've broken their promise. (In English, a sentence that is *false* is called a *broken promise* or a *lie*!) However, suppose you get a 'B'. Your parents know you worked really, really hard and decide to take you out anyway. Now, did they break their promise? Of course not. They promised that if you DO get an 'A', then they'll take you out. Their promise didn't give any information about what might happen if you *don't* get an 'A'.

In both of the preceding examples, it appears that in order for the sentence '*If A, then B*' to be TRUE, it must be that whenever A is true, B must also be true.

a true mathematical sentence of the form 'If A, then B'

Here's a very simple mathematical implication:

$$\text{If } x = 2, \text{ then } x^2 = 4$$

Use your intuition: would you want to say that this sentence is true or false? Based on English sentences of the same form, you'd probably want to say that it is true: because whenever the sentence $x = 2$ is true, then the sentence $x^2 = 4$ is also true. The only number that makes '$x = 2$' true is 2, and 2^2 is equal to 4.

Indeed, to a mathematician, the sentence

$$\text{If } x = 2, \text{ then } x^2 = 4$$

is true. This is because WHENEVER the sentence $x = 2$ is true, the sentence $x^2 = 4$ is also true.

★★

implicit universal quantifier

Really, the sentence

$$\text{If } x = 2, \text{ then } x^2 = 4$$

is a (true) implicit generalization,

$$\text{For all } x, \text{ if } x = 2 \text{ then } x^2 = 4.$$

However, it is common usage to leave the universal quantifier 'For all' implicit, rather than explicit.

a false mathematical sentence of the form 'If A, then B'

Now consider the implication:

$$\text{If } x^2 = 4, \text{ then } x = 2$$

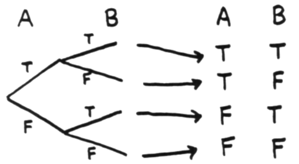
Where does your intuition lead you? Would you want to call this sentence *true* or *false*? In order to be true, you probably want to be assured that WHENEVER $x^2 = 4$ is true, then $x = 2$ must also be true. This is not the case. It is possible to choose x such that $x^2 = 4$ is true, but $x = 2$ is false. Just choose x to be -2. Then, $(-2)^2 = 4$ is true, but $-2 = 2$ is false. Thus, the mathematical sentence

$$\text{If } x^2 = 4, \text{ then } x = 2$$

is false.

the truth table for the sentence If A, then B

Now let's investigate the mathematical sentence '*If A then B*' even more precisely. The *sub-sentences* A and B can be true or false, and there are four possible combinations:



It is conventional to give the truth values of the sentence '*If A, then B*' by using a *truth table*:

A	B	If A, then B
T	T	T
T	F	F
F	T	T
F	F	T

the only way the sentence is FALSE, is if A is TRUE, and B is FALSE

the sentence is TRUE

Look at the truth table. Under what conditions is the sentence *If A, then B* TRUE? Well, it is true if A is true, and B is true. The first line of the truth table tells us this. This is not surprising.

The sentence '*If A, then B*' is also true if A is false. The third and fourth lines of the truth table tell us this. This is in perfect harmony with the English usage. If your parents say,

If you get an 'A' in calculus, then we'll buy you a dinner.

and then buy you a dinner when you get a 'B', they didn't lie. Their statement was still true.

'hypothesis' and 'conclusion' of an implication

In the sentence '*If A, then B*', A is called the *hypothesis* (of the implication) and B is called the *conclusion* (of the implication).

The third and fourth lines of the truth table tell us that if the hypothesis of an implication is false, then the sentence '*If A, then B*' is automatically true.

IMPORTANT!

To check that an implication is true, one need only check that whenever A is true, so is B

Here's an extremely important consequence of the definition of the sentence '*If A, then B*'. To see if a sentence of this form is TRUE, we need only verify that whenever A is true, so is B. We don't bother to check what happens if A is false: because if A is false, the sentence is automatically true.

EXERCISE 6

the word 'hypothesis' in mathematics

♣ We have run across the word *hypothesis* a few times now. We talked earlier about the *hypotheses* of a theorem; now we have the *hypothesis* of an implication. Does it make sense to use the same word in both situations? Comment.

An alternate form of the sentence 'If A, then B'; $A \implies B$

Since implications are extremely common in mathematics, it should not be surprising that there is more than one way to say the same thing. The sentence '*If A, then B*' can also be written in the form

$$A \implies B$$

and read as '*A implies B*'.

The next example gives some practice with implications.

EXAMPLE

practice with implications

Determine if the following implications are TRUE or FALSE. If an implication involves a variable, then in order to be true, it must be true for all possible choices of the variable.

- If $2 = 1$, then $2 = 5$

This sentence is true. Here, A is false, B is false, and '*If A, then B*' is true (line 4 of the truth table). Whenever the hypothesis of an implication is false, the implication is automatically true. Some students have trouble with this: they can't believe that a sentence can be *true* with so much *false* stuff floating around!

- $x > 2 \implies x > 1$

This sentence is true. Whenever x is a number greater than 2, then it is also greater than 1. That is, whenever $x > 2$ is true, then $x > 1$ must also be true. Note that this sentence is true for ALL real numbers x. In particular, if x is 1, then the sentence

$$1 > 2 \implies 1 > 1$$

is automatically true, because the hypothesis is false.

- If $x > 1$, then $x > 2$

This sentence is false. It is possible to make $x > 1$ true, but $x > 2$ false. Choose, say, $x = 1.5$.

showing that an implication is false

If an implication involves a variable, then to show that it is FALSE, you must produce a *specific* choice for the variable that makes the hypothesis TRUE, but the conclusion FALSE. Here's the format to use in such a situation:

Problem: TRUE or FALSE: $x > 1 \quad \implies \quad x > 2$

Solution: FALSE. Let $x = 1.5$. Then, the hypothesis

$$1.5 > 1$$

is true, but the conclusion

$$1.5 > 2$$

is false.

counterexample

A specific choice of variable(s) for which a sentence is false is called a *counterexample.*

Problem: Decide if the sentence 'If $x > 1$, then $x > 2$' is true or false. If false, give a counterexample.

Solution: The sentence is false. Let $x = 1.9$. Then, the hypothesis $1.9 > 1$ is true, but the conclusion $1.9 > 2$ is false.

EXAMPLE

Problem: Is the sentence

$$y^2 = 9 \implies y = 3$$

true or false? If false, give a counterexample.

Solution: The sentence is false. Let $y = -3$. Then, the hypothesis $(-3)^2 = 9$ is true, but the conclusion $-3 = 3$ is false.

EXERCISE 7

Decide if the following mathematical sentences are true or false. If false, give a counterexample, using the form illustrated above.

♣ 1. If $x = 3$, then $x^2 = 9$

♣ 2. If $x^2 = 9$, then $x = 3$

♣ 3. $x = 2 \implies |x| = 2$

♣ 4. If $|x| = 2$, then $x = 2$

♣ 5. $a < b \implies |a| < |b|$

♣ 6. If $0 < a < b$, then $|a| < |b|$

EXERCISE 8

For this entire exercise, assume that the sentence $A \implies B$ is TRUE.

♣ 1. What (if anything) can you conclude about the truth value of A?

♣ 2. What (if anything) can you conclude about the truth value of B?

♣ 3. Suppose you know that A is true. What (if anything) can you conclude about the truth value of B?

♣ 4. Suppose you know that B is true. What (if anything) can you conclude about the truth value of A?

♣ 5. Suppose you know that A is false. What (if anything) can you conclude about the truth value of B?

♣ 6. Suppose you know that B is false. What (if anything) can you conclude about the truth value of A?

QUICK QUIZ
sample questions

1 Give a precise statement of the Intermediate Value Theorem.

2 Suppose that f is continuous on the interval $[1, 3]$; $f(1)$ is negative, and $f(3)$ is positive. Must the function f take on the value 0 on $[1, 3]$? Why or why not?

3 TRUE or FALSE: $1 = 2 \implies 3 = 4$.

4 TRUE or FALSE: If $|x| = 1$, then $x = 1$. If the sentence is false, give a counterexample.

5 Give the truth table for the mathematical sentence $A \implies B$.

KEYWORDS
for this section

The Intermediate Value Theorem, use of the word 'between', using the IVT to guarantee and estimate solutions to equations. Implications: notation, truth table, hypothesis, conclusion, counterexample.

END-OF-SECTION EXERCISES

Determine if the following implications are TRUE or FALSE. If false, give a counterexample. The context will determine if the variable(s) used are numbers, functions, sentences, or sets.

Remember that if a sentence involves a variable, then to be TRUE, the sentence must be true for all possible choices of the variable.

1. If f is continuous on $[a,b]$ and D is any number between $f(a)$ and $f(b)$, then there exists a number d between a and b with $f(d) = D$.
2. If f is continuous on $[0,2]$, $f(0) = 1$, and $f(2) = 4$, then there exists $d \in (0,2)$ with $f(d) = 3$.
3. If A is false, then the sentence $A \implies B$ is true.
4. If B is false, then the sentence $A \implies B$ is false.
5. If B is true, then the sentence 'If A, then B' is true.
6. If A is true, then the sentence 'If A, then B' is true.
7. If $|t| = 0$, then $t = 0$
8. If $|t| = 1$, then $t = 1$
9. If $t = 1$, then $|t| = 1$
10. If $t = -1$, then $|t| = 1$