4.6 Differentiating Products and Quotients

Introduction

The derivative of a sum of differentiable functions is always the sum of the derivatives. Is the derivative of a *product* of differentiable functions always the product of the derivatives? The next example shows that the answer is NO.

EXAMPLE
the derivative of a product is NOT the product of the derivatives

Let $f(x) = x$ and $g(x) = x^2$. Then, the product function fg is defined by

$$(fg)(x) := f(x) \cdot g(x) = x^3 \ ,$$

and has derivative $(fg)'(x) = 3x^2$. However, the product of the derivatives is $f'(x) \cdot g'(x) = (1) \cdot (2x) = 2x$. Note that:

$$(fg)'(x) \neq f'(x) \cdot g'(x)$$

EXERCISE 1
the derivative of a quotient is NOT the quotient of the derivatives

♣ Find a pair of functions f and g for which:

$$(\frac{f}{g})'(x) \neq \frac{f'(x)}{g'(x)}$$

The *correct* rule to be used for differentiating products is called the *Product Rule for Differentiation*, and is stated next. Since it is such a surprising result, it is absolutely necessary to study the proof carefully, to understand what gives rise to this formula!

PRODUCT RULE for differentiation

Suppose that f and g are both differentiable at x. Then, the product function fg is also differentiable at x, and:

$$(fg)'(x) = f(x)g'(x) + f'(x)g(x)$$

In words, *the derivative of a product is the first times the derivative of the second, plus the derivative of the first times the second.*

The Product Rule is also commonly written as:

$$\frac{d}{dx} f(x)g(x) = f(x)g'(x) + f'(x)g(x)$$

The trick in the proof of the Product Rule is to *add zero* in an appropriate form, in order to rewrite the difference quotient for fg in a way that brings the difference quotients for f and g into the picture. Study the following proof:

PROOF
of the Product Rule

Proof. Let f and g be differentiable at x, and let $(fg)(x) = f(x)g(x)$. Then $(fg)(x+h) = f(x+h)g(x+h)$. The lines below are numbered for easy reference:

$$
\begin{aligned}
&(1)\ (fg)'(x) := \lim_{h\to 0} \frac{(fg)(x+h) - (fg)(x)}{h} \\
&(2) \qquad\qquad = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&(3) \qquad\qquad = \lim_{h\to 0} \frac{f(x+h)g(x+h) \overbrace{- f(x+h)g(x) + f(x+h)g(x)}^{\text{add } 0} - f(x)g(x)}{h} \\
&(4) \qquad\qquad = \lim_{h\to 0} f(x+h)\frac{g(x+h) - g(x)}{h} + \frac{f(x+h) - f(x)}{h}g(x) \\
&(5) \qquad\qquad = \lim_{h\to 0} f(x+h)\frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}g(x) \\
&(6) \qquad\qquad = \lim_{h\to 0} f(x+h) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} + \left(\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right) g(x) \\
&(7) \qquad\qquad = f(x)g'(x) + f'(x)g(x) \ \blacksquare
\end{aligned}
$$

a discussion of each line of the previous proof; lines (1)–(5)

You must understand the previous proof. In particular, it is essential that you understand the justification for *each step* in this proof.

In line (1), the definition of the derivative is used, applied to the function fg.

In line (2), the definition of the function fg is used.

In line (3), the number 0 is added in an appropriate form. The motivation for adding zero in this form is to bring the difference quotients for f and g into the picture!

In line (4), $f(x+h)$ is factored out of the first two terms; and $g(x)$ is factored out of the last two terms.

In line (5), the limit of a sum is written as the sum of the limits. Is this allowable? Only if the individual limits

$$\lim_{h\to 0} f(x+h)\frac{g(x+h) - g(x)}{h} \quad \text{and} \quad \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}g(x)$$

exist! Do they? The next few steps show that, indeed, they do.

line (6)

In the first part of line (6), the limit of a product is written as the product of the limits. Again, this is allowable only if each individual limit

$$\lim_{h\to 0} f(x+h) \quad \text{and} \quad \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

exists. We see in the next step that, indeed, these limits DO exist!

In the second part of line (6), one notes that $g(x)$ is *constant* relative to the limit being investigated. That is, $g(x)$ has nothing to do with h; and constants can be 'slid out' of the differentiation process.

line (7)

In line (7), it is finally demonstrated that all the individual limits exist, thus justifying, (after the fact), the limit operations used in the previous few steps. Both $\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$ and $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exist and equal, respectively, $g'(x)$ and $f'(x)$, by the hypothesis that f and g are differentiable at x.

Why is $\lim_{h\to 0} f(x+h) = f(x)$?

Why is

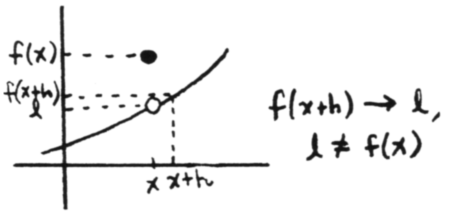
$$\lim_{h\to 0} f(x+h) = f(x)\ ?$$

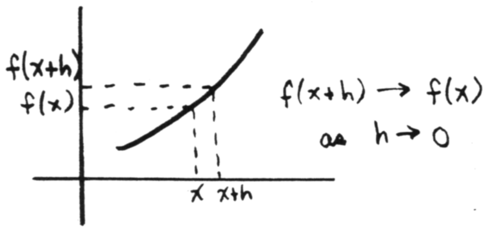
Note that this (true) sentence states that as h approaches 0, the numbers $f(x+h)$ approach $f(x)$. But when h is close to 0, $x+h$ is close to x; so, rephrasing, when the inputs to f are close to x, the corresponding outputs are close to $f(x)$. *This is precisely what it means for f to be continuous at x!* Indeed,

$$f \text{ is continuous at } x \quad \iff \quad \lim_{h\to 0} f(x+h) = f(x)\ ,$$

so that the statement '$\lim_{h\to 0} f(x+h) = f(x)$' gives an equivalent characterization of continuity of f at x.

But how do we know that f is continuous at x? By hypothesis, f is differentiable at x. Since differentiability is a stronger condition than continuity, f must also be continuous at x.





EXERCISE 2
prove the product rule

♣ Prove the product rule for differentiation, *without looking at the book.* Be sure to justify each step in your proof. It would not be unreasonable for your instructor to ask you to prove the product rule on an in-class exam.

'testing' the Product Rule

When presented with a new result, it is always a good idea to 'test it out' in a situation where you already know the answer, to gain confidence. Therefore, we first differentiate a product whose derivative can be found by other means, so that we have a way to 'check' the answer derived from the product rule.

Problem: Let $f(x) = (x^2+1)(2x+5)$. Find f' by
(a) multiplying f out, and differentiating term-by-term; and
(b) using the product rule.

Solution:

(a) $f(x) = 2x^3 + 5x^2 + 2x + 5$, so:

$$f'(x) = 6x^2 + 10x + 2$$

(b) Using the product rule:

$$\begin{aligned} f'(x) &= (x^2+1)(2) + (2x)(2x+5) \\ &= 2x^2 + 2 + 4x^2 + 10x \\ &= 6x^2 + 10x + 2 \end{aligned}$$

Compare!

EXERCISE 3

♣ Let $f(x) = (x+4)(2x^3-1)$. Find f' in two ways: by multiplying out and differentiating term-by-term, and by using the product rule. Compare your results.

EXAMPLE

using the Product Rule

Problem: Differentiate $f(x) = x\sqrt{3x+2}$.

Solution: Note carefully how the $\frac{d}{dx}$ operator is used for intermediate steps in the solution that follows:

$$\begin{aligned} f'(x) &= x\frac{d}{dx}(3x+2)^{1/2} + (1)\sqrt{3x+2} && \text{(product rule)} \\ &= x \cdot \frac{1}{2}(3x+2)^{-1/2}(3) + \sqrt{3x+2} && \text{(general power rule)} \\ &= \frac{3x}{2\sqrt{3x+2}} + \sqrt{3x+2} && \text{(simplify)} \end{aligned}$$

It is sometimes desirable to write the formula for f' in a way that has no radicals in denominators. This is accomplished by rationalizing the denominator in the first term. The word 'rationalize' means to 'remove the radical'. Thus, to 'rationalize the denominator' means to 'remove the radical in the denominator'. Rewriting the first term yields

$$\begin{aligned} \frac{3x}{2\sqrt{3x+2}} &= \frac{3x}{2\sqrt{3x+2}} \cdot \frac{\sqrt{3x+2}}{\sqrt{3x+2}} \\ &= \frac{3x\sqrt{3x+2}}{2(3x+2)}\ , \end{aligned}$$

so that $f'(x)$ becomes:

$$f'(x) = \frac{3x\sqrt{3x+2}}{2(3x+2)} + \sqrt{3x+2}$$

The result can be expressed as a single term by getting a common denominator and combining fractions:

$$\begin{aligned} f'(x) &= \frac{3x\sqrt{3x+2}}{2(3x+2)} + \sqrt{3x+2} \cdot \frac{2(3x+2)}{2(3x+2)} \\ &= \frac{\sqrt{3x+2}(3x+2(3x+2))}{2(3x+2)} \\ &= \frac{\sqrt{3x+2}(9x+4)}{2(3x+2)} \end{aligned}$$

EXAMPLE
using the Product Rule

Problem: Differentiate $f(x) = 3x^2 \cdot h(2x-1)$, where h is a differentiable function of one variable.

Solution: Observe that $h(2x-1)$ denotes the function h, acting on the input $2x-1$, and NOT h times $2x-1$. Thus, $h(2x-1)$ is a composition of functions, which is differentiated using the chain rule. The *overall* form of the function f being differentiated is a product: it is the function $3x^2$, multiplied by $h(2x-1)$. Thus, the product rule is first applied; note how the $\frac{d}{dx}$ operator is conveniently used for intermediate steps:

$$\begin{aligned} f'(x) &= 3x^2 \cdot \frac{d}{dx}\big(h(2x-1)\big) + 6x \cdot h(2x-1) && \text{(product rule)} \\ &= 3x^2 \cdot h'(2x-1) \cdot 2 + 6x \cdot h(2x-1) && \text{(chain rule)} \\ &= 6x^2 \cdot h'(2x-1) + 6x \cdot h(2x-1) && \text{(simplify)} \end{aligned}$$

This expression for f' *cannot* be simplified further, unless we are given additional information about the function h.

EXAMPLE
generalizing the product rule

Problem: Find $\frac{d}{dx}a(x)b(x)c(x)$. Assume that a, b, and c are differentiable.

Solution: Use the 'treat it as a singleton' trick!

$$\begin{aligned} \frac{d}{dx}a(x)b(x)c(x) &= \frac{d}{dx}[\big(a(x)b(x)\big) \cdot c(x)] && \text{(group)} \\ &= a(x)b(x) \cdot c'(x) + \left(\frac{d}{dx}a(x)b(x)\right) \cdot c(x) && \text{(product rule)} \\ &= a(x)b(x)c'(x) + \big[a(x)b'(x) + a'(x)b(x)\big]c(x) && \text{(product rule)} \\ &= a(x)b(x)c'(x) + a(x)b'(x)c(x) + a'(x)b(x)c(x) && \text{(multiply out)} \\ &= a'(x)b(x)c(x) + a(x)b'(x)c(x) + a(x)b(x)c'(x) && \text{(rearrange)} \end{aligned}$$

Observe the pattern? By defining the function abc via the rule

$$(abc)(x) := a(x)b(x)c(x) \ ,$$

this result can be written as:

$$(abc)'(x) = a'(x)b(x)c(x) + a(x)b'(x)c(x) + a(x)b(x)c'(x)$$

It can also be shown that, (suppressing the '(x)', for convenience):

$$(abcd)' = a'bcd + ab'cd + abc'd + abcd'$$

The 'regular' product rule of course also follows this pattern:

$$(fg)' = f'g + fg'$$

These generalized results are extremely useful when differentiating products with more than two factors.

EXERCISE 4

♣ 1. Make a conjecture (educated guess) about the formula for $(abcde)'$. Assume that a, b, c, d and e are all differentiable. Feel free to suppress the '(x)' in your answer.

♣ 2. Use a 'generalized' product rule to differentiate $f(x) = (2x+1)(x^2 - 3)(4-x)$.

EXAMPLE

differentiating a quotient

Problem: Use any available differentiation tools to find $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)$.

Solution:

$$\begin{aligned}
\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) &= \frac{d}{dx}\big(f(x)\cdot\frac{1}{g(x)}\big) && \text{(rewrite as a product)}\\
&= f(x)\frac{d}{dx}(g(x))^{-1} + f'(x)\cdot\frac{1}{g(x)} && \text{(product rule)}\\
&= f(x)\cdot(-1)(g(x))^{-2}g'(x) + \frac{f'(x)}{g(x)} && \text{(general power rule)}\\
&= \frac{-f(x)g'(x)}{(g(x))^2} + \frac{f'(x)}{g(x)}\cdot\frac{g(x)}{g(x)} && \text{(simplify, get common denom.)}\\
&= \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} && \text{(simplify)}
\end{aligned}$$

The formula just derived tells us how to differentiate a quotient of functions, and is called the *Quotient Rule for Differentiation.* Note that the derivative of a quotient is *not* the quotient of the derivatives! That is:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) \neq \frac{f'(x)}{g'(x)}$$

(This should not be a surprise; a quotient is a special kind of product—and we already learned that the derivative of a product is NOT the product of the derivatives.) A precise statement of the quotient rule follows:

QUOTIENT RULE for differentiation

Suppose that f and g are both differentiable at x, and $g(x) \neq 0$. Then, the function $\frac{f}{g}$ is also differentiable at x, and:

$$\big(\frac{f}{g}\big)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

In words, the derivative of a quotient is: *the bottom, times the derivative of the top, minus the top, times the derivative of the bottom, all over the bottom squared.*

memory device for the quotient rule

Some students find the following cute memory device helpful:

$$\frac{d}{dx}\big(\frac{HI}{HO}\big) = \frac{HO\,dHI - HI\,dHO}{HO\,HO}$$

(Note that 'HI' is 'high up' on the fraction.)

EXAMPLE
using the quotient rule

Problem: Differentiate $f(x) = \frac{2}{x}$ in two ways: using the simple power rule, and using the quotient rule.

Solution: Using the simple power rule:

$$f(x) = 2x^{-1}$$
$$f'(x) = -2x^{-2} = \frac{-2}{x^2}$$

Using the quotient rule:

$$f'(x) = \frac{x(0) - 2(1)}{x^2} = \frac{-2}{x^2}$$

EXERCISE 5

Differentiate each of the following functions in two ways: using the quotient rule, and NOT using the quotient rule. Compare your answers.

♣ 1. $f(x) = \frac{x}{2x-1}$

♣ 2. $g(x) = \frac{3}{(1-x)^4}$

return to the Simple Power Rule;
proving that $\frac{d}{dx}x^n = nx^{n-1}$ *for positive integers* n

Now that the product rule is in hand, it is possible to give an easy proof of the Simple Power Rule for Differentation in the case where n is a positive integer. An extremely important technique, called *proof by induction*, is used. This technique of *proof by induction* is discussed next.

Proof by Induction

A standard approach to proving that a formula is true *for all positive integers* is to use a *proof by induction*. The logic involved in this sort of proof is sometimes called *the domino principle*:

- STEP 1: First, show that the formula is true when $n = 1$.
- STEP 2: Next, show that whenever the formula is true for a positive integer K, then it must also be true for the next positive integer $K + 1$.

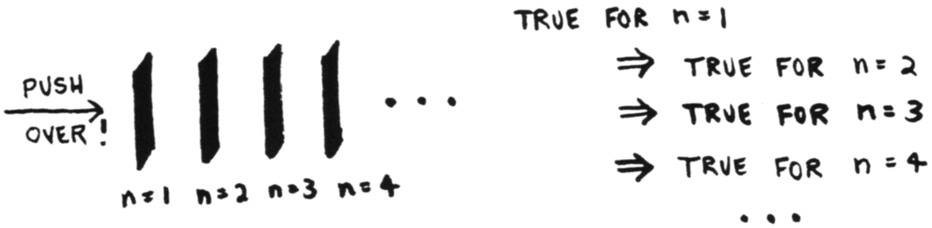
If both of these steps can be accomplished, then look what happens:

Since the formula is true for $n = 1$, it must also be true for $n = 2$.

Since the formula is true for $n = 2$, it must also be true for $n = 3$.

Since the formula is true for $n = 3$, it must also be true for $n = 4$.

- STEP 3: Continuing this scheme, conclude that the formula must be true for *all* positive integers!



Proof that $\frac{d}{dx}x^n = nx^{n-1}$ *for all positive integers* n

Problem: Prove that $\frac{d}{dx}x^n = nx^{n-1}$ for all positive integers n.

Solution: Use a proof by induction.

- Step 1: Show that the formula $\frac{d}{dx}x^n = nx^{n-1}$ is true when $n = 1$.

  Solution to Step 1: When $n = 1$, $x^n = x^1 = x$ and $nx^{n-1} = (1)x^{1-1} = 1$. Since indeed $\frac{d}{dx}x = 1$, the formula is true when $n = 1$.

- Step 2: Let K be any positive integer, and *assume* that the formula is true when $n = K$. (This assumption is commonly referred to as the *inductive hypothesis*.) Show that it must also be true when $n = K + 1$.

  Solution to Step 2: Let K be a positive integer, and assume that $\frac{d}{dx}x^K = Kx^{K-1}$. It is now necessary to show that the formula holds when $n = K+1$. Each of the lines below is numbered for easy reference:

$$
\begin{aligned}
&(1) \qquad & \frac{d}{dx}x^{K+1} &= \frac{d}{dx}x \cdot x^K \\
&(2) & &= x(Kx^{K-1}) + (1)x^K \\
&(3) & &= Kx^K + x^K \\
&(4) & &= (K+1)x^K \\
&(5) & &= (K+1)x^{(K+1)-1}
\end{aligned}
$$

lines (1)–(3)

In line (1), x^{K+1} is viewed as the product $x \cdot x^K$ so that the product rule can be applied.

In line (2), the product rule is used to differentiate $x \cdot x^K$. Note that the *inductive hypothesis* $\frac{d}{dx}x^K = Kx^{K-1}$ is used to differentiate x^K.

In line (3), the expression is simplified, using the fact that $x \cdot x^{K-1} = x^1 x^{K-1} = x^{1+K-1} = x^K$.

lines (4) and (5)

In line (4), x^K is factored out of each term.

In line (5), one notes that $K = (K+1) - 1$. Equating line (1) to line (5), we see that

$$\frac{d}{dx}x^{K+1} = (K+1)x^{(K+1)-1} \ ,$$

so that the formula

$$\frac{d}{dx}x^n = nx^{n-1}$$

holds when n is replaced by $K + 1$. This completes Step (2).

- Step 3: Conclude that the result holds for *all* positive integers n.

  Solution to Step 3: Therefore, $\frac{d}{dx}x^n = nx^{n-1}$ for all positive integers n. ∎

EXERCISE 6

♣ 1. Use a *proof by induction* to prove that the formula

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

holds for all positive integers n. Be sure to clearly indicate where the inductive hypothesis is used.

Hint: To show that the formula holds for $n = K + 1$, you must show that:

$$1 + 2 + \ldots + K + (K+1) = \frac{(K+1)(K+2)}{2}$$

♣ 2. Use the previous formula to find:

$$1 + 2 + 3 + \ldots + 512$$

♣ 3. How could the formula be used to find

$$100 + 101 + \ldots + 512 \text{ ?}$$

More important differentiation tools have been added in this section. The list is now complete, and is given below:

DIFFERENTIATION TOOLS

prime notation	$\frac{d}{dx}$ operator
if $f(x) = k$, then $f'(x) = 0$	$\frac{d}{dx}(k) = 0$
$(kf)'(x) = k \cdot f'(x)$	$\frac{d}{dx}\big(kf(x)\big) = k \cdot f'(x)$
$(f+g)'(x) = f'(x) + g'(x)$	$\frac{d}{dx}\big(f(x) + g(x)\big) = f'(x) + g'(x)$
$(f-g)'(x) = f'(x) - g'(x)$	$\frac{d}{dx}\big(f(x) - g(x)\big) = f'(x) - g'(x)$
if $f(x) = x^n$, then $f'(x) = nx^{n-1}$	$\frac{d}{dx}(x^n) = nx^{n-1}$
if $f(x) = (g(x))^n$, then $f'(x) = n(g(x))^{n-1} \cdot g'(x)$	$\frac{d}{dx}(g(x))^n = n(g(x))^{n-1} \cdot g'(x)$
if $f(x) = e^x$, then $f'(x) = e^x$	$\frac{d}{dx}(e^x) = e^x$
if $f(x) = e^{g(x)}$, then $f'(x) = e^{g(x)} \cdot g'(x)$	$\frac{d}{dx}(e^{g(x)}) = e^{g(x)} \cdot g'(x)$
if $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$	$\frac{d}{dx}(\ln x) = \frac{1}{x}$
if $f(x) = \ln(g(x))$, then $f'(x) = \frac{1}{g(x)} \cdot g'(x)$	$\frac{d}{dx}\big(\ln g(x)\big) = \frac{1}{g(x)} \cdot g'(x)$
$(fg)'(x) = f(x)g'(x) + f'(x)g(x)$	$\frac{d}{dx}\big(f(x)g(x)\big) = f(x)g'(x) + f'(x)g(x)$
$(abc)' = a'bc + ab'c + abc'$	$\frac{d}{dx}(abcd) = a'bcd + ab'cd + abc'd + abcd'$
$(\frac{f}{g})'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$	$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

QUICK QUIZ
sample questions

1 Give a precise statement of the Product Rule for Differentiation.
2 Give a precise statement of the Quotient Rule for Differentiation.
3 Differentiate: $f(x) = x(x+1)^5$
4 Differentiate: $f(x) = \frac{2x+1}{e^{2x}}$
5 Differentiate: $y = x(x+1)(x^2+1)$ Use any correct method.

KEYWORDS
for this section

The product rule for differentiation, proof of the product rule, 'generalized' product rules, the quotient rule for differentiation, proof by induction.

END-OF-SECTION EXERCISES

The purpose of these exercises is to give you additional practice with all the differentiation formulas and notation.

♣ Differentiate the following functions. Use any appropriate formulas. Answer all the additional questions. If an object does not exist, so state. Be sure to write complete mathematical sentences.

1 $y = x(2-x)^3$; find $y(0)$, $y(t^2)$, $y'(0)$, and $y'(t)$

2 $y = e^{-x}\sqrt{x^2-1}$; find $y(1)$, $y'(1)$, $\frac{dy}{dx}\big|_{x=\sqrt{2}}$, and $y(1)\cdot y'(\sqrt{2})$

3 $f(x) = e^x \ln x$; find $\mathcal{D}(f)$, $\mathcal{D}(f')$, $f'(e^x)$, and $f'(e^2)$

4 $f(x) = \ln(\ln x)$; find $\mathcal{D}(f)$, $\mathcal{D}(f')$, $f'(e^x)$, and $f'(f(e))$

5 $g(x) = e^{(e^x)}$; find $\lim_{x\to 0} g(x)$, $\lim_{x\to 0} g'(x)$, $\mathcal{D}(g)$, $g(g'(g(0)))$

6 $g(x) = (x-1)(2x+1)(1-x)^7$; find $\lim_{x\to 0} g(x)$, $\lim_{t\to 0} g(t)$, $\lim_{x\to 0} g'(x)$, and $g'(0)$. Is g' continuous at 0? Why or why not?

7 $h(x) = \ln\left(\frac{e^x}{x+1}\right)$; find the equation of the tangent line to the graph of h at $x = 0$

8 $h(x) = \sqrt{\ln x^3}$; find $\mathcal{D}(h)$ and the equation of the tangent line to the graph of h at $x = e$

9 $f(x) = e^{2x}(2x+1)^7$; find the equation of the tangent line to the graph of f at $x = 0$

10 $g(x) = (ax+b)^2(cx+d)^3(x+1)^4$. Assume that a, b, c and d are constants.

11 $h(t) = \frac{e}{(3t-1)^4}$; find the equation of the tangent line to the graph of h at $t = \frac{2}{3}$

12 $y = \frac{\ln t}{\sqrt{t+2}}$; what is the instantaneous rate of change of y with respect to t when $t = 1$?

13 Find all points on the graph of $y = [(x-3)(x+1)(2x-1)]^2$ where the tangent line is horizontal. (Hint: There are 5 such points.)