5.3 The Second Derivative; Inflection Points

the second derivative function, f''

If a function f is sufficiently smooth, then we can differentiate once to get f', and again to get f''. The function f'' is called the *second derivative of* f, and is by far the most important higher-order derivative.

Recall that at $x = c$:

$$g'(c) \text{ gives the instantaneous rate of change of } g(x) \text{ with respect to } x$$

Taking $g = f'$:

$$(f')'(c) = f''(c) \text{ gives the instantaneous rate of change of } f'(x) \text{ with respect to } x$$

Since the numbers $f'(x)$ give the slopes of the tangent lines to the graph of f, the function f'' tells *how fast the slopes of the tangent lines to the graph of f are changing.*

concave up

For example, if the slopes of the tangent lines are *increasing*, then the scenario is the following:

Thus, when $f''(x) > 0$, the shape illustrated above ('holding water') is generated. Such a graph is said to be *concave up.* Observe that the more positive $f''(x)$ is, the more quickly the slopes of the tangent lines increase, and hence the more rapidly the graph turns.

concave down

Similarly, if the slopes of the tangent lines are *decreasing*, then the scenario is the following:

Thus, when $f''(x) < 0$, the shape illustrated above ('shedding water') is generated. Such a graph is said to be *concave down.* Again note that the more negative $f''(x)$ is, the more quickly the slopes of the tangent lines decrease, and hence the more rapidly the graph turns.

EXERCISE 1

Sketch graphs satisfying the following properties:

♣ 1. $f(1) = f(3) = 2$, $f(2) = 3$; the slopes of the tangent lines increase on the interval $(1, 2)$ and decrease on $(2, 3)$; f is continuous at $x = 2$

♣ 2. $f(x) < 0$ on $[1, 3]$; the slopes of the tangent lines decrease on the interval $(1, 2)$ and increase on $(2, 3)$

The precise definitions of *concave up* and *concave down* on an interval follow.

DEFINITION

concave up on I
concave down on I

Let I be an interval of real numbers.

If $f''(x) > 0$ for all x in I, then f is *concave up* on I.

If $f''(x) < 0$ for all x in I, then f is *concave down* on I.

a common convention concerning DEFINITIONS

Every definition is, either implicitly or explicitly, a statement of *equivalence*. For this reason, there is a convention regarding definitions: they may be stated as 'If ... then ...' sentences, when in actuality they are 'if and only if' sentences.

Without this convention, the previous definition would have to be written something like this:

DEFINITION

concave up on I
concave down on I

Let I be an interval of real numbers.

f is *concave up* on I if and only if, for all $x \in I$, $f''(x) > 0$.

f is *concave down* on I if and only if, for all $x \in I$, $f''(x) < 0$.

EXAMPLE

concavity of a line

Consider the linear function $f(x) = ax + b$. As one moves from point to point, the slopes of the tangent lines *do not change at all*. This information is reflected in the second derivative:

$$\begin{aligned} f(x) = ax + b &\implies f'(x) = a \\ &\implies f''(x) = 0 \end{aligned}$$

Conversely, suppose a function g has the property that $g''(x) = 0$. Then it must be that $g'(x) = a$ for some real number a. (♣ Why?) But then, g must be a line with slope a, so that $g(x) = ax + b$ for some real number b.

The process of going from information about *derivatives* back to information about the *original function* is called *integration* or *antidifferentiation*, and is discussed in more detail in later sections.

NOTE about the word: 'Conversely'

Recall that the sentence $A \Rightarrow B$ is called an *implication*. The new sentence $B \Rightarrow A$ is called the *converse* of the sentence $A \Rightarrow B$.

It has been seen that the truth of $A \Rightarrow B$ in no way influences the truth of $B \Rightarrow A$. Each sentence must be investigated separately. If you have just investigated the sentence $A \Rightarrow B$, and now want to investigate its converse, $B \Rightarrow A$, it is common to say: *Conversely, ...* . This prepares the reader for the fact that you are about to investigate the converse.

EXERCISE 2

Sketch graphs satisfying the following properties:

♣ 1. $f(1) = 1$, $f'(3) = \frac{1}{2}$, and $f''(x) = 0 \;\; \forall\, x \in [1, 5]$. What must $f(5)$ be?

♣ 2. $f(3) = 4$, $f'(3) = 2$, $f''(x) = 0$ on $(1, 5)$, f is continuous at $x = 5$, f is defined but not continuous at $x = 1$

EXERCISE 3

Write down the *converse* of each of the following implications, using any correct notation. Is the original implication true? What about its converse?

♣ 1. If $x = 2$, then $x^2 = 4$

♣ 2. $1 = 2 \Rightarrow 1 + 1 = 2$

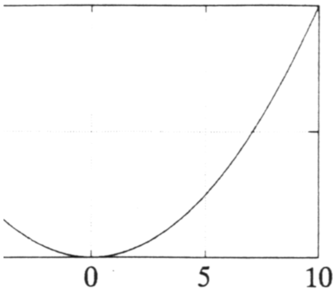
♣ 3. If $1 = 2$, then $2 = 3$

♣ 4. Now, let $A \Rightarrow B$ be an implication. What is the converse of the converse? What is the contrapositive of the converse? What is the converse of the contrapositive?

EXAMPLE

concavity of the squaring function

Consider the function $f(x) = x^2$. Here, $f'(x) = 2x$ and $f''(x) = 2$. The second derivative is constantly 2. The curve always 'turns' at exactly the same rate. Thus, at every point on the graph of f, the slopes of the tangent lines are changing twice as fast as the x value of the point is changing. Thus, when x changes by 1, one should find that the slope of the tangent line changes by 2 .



For example, consider the point $(0, f(0)) = (0, 0)$ on the graph of f. Here, the tangent line has slope $f'(0) = 2 \cdot 0 = 0$. Move one unit to the right, to the point $(1, f(1)) = (1, 1)$. Here, the tangent line has slope $f'(1) = 2 \cdot 1 = 2$. When x increased by 1, the slope of the tangent line increased by 2 !

Let's investigate this fact in more generality. Let $(x, f(x))$ be *any* point on the graph of f. Then, $(x+1, f(x+1))$ is the point with x value increased by 1 . The slope of the tangent line at $(x, f(x))$ is $f'(x) = 2x$. The slope of the tangent line at $(x + 1, f(x + 1))$ is:

$$\begin{aligned} f'(x+1) &= 2(x+1) \\ &= 2x + 2 \\ &= f'(x) + 2 \end{aligned}$$

$f(x) = x^2$

CONSTANT "TURNING RATE": $f''(x) = 2$

Thus, when x increases by 1, the slope of the tangent line increases by 2. That is, when $\Delta x = 1$, we have:

$$\begin{aligned} \Delta f' &= f'(x+1) - f'(x) \\ &= (f'(x) + 2) - f'(x) \\ &= 2 \end{aligned}$$

EXERCISE 4

Consider the function $f(x) = 2x^2$.

♣ 1. When x changes by an amount Δx, how much do you expect the slopes of the tangent lines to change by?

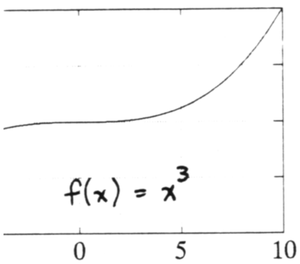
♣ 2. Find the slope of the tangent line at the point $(x + \Delta x, f(x + \Delta x)$ and compare it to the slope of the tangent line at $(x, f(x))$.

♣ 3. What is $\Delta f'$?

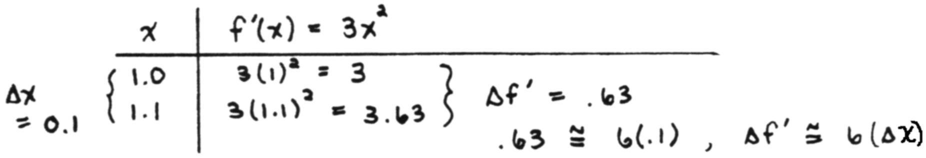
EXAMPLE

concavity of the cubing function

Now consider the function $f(x) = x^3$. Here, $f'(x) = 3x^2$ and $f''(x) = 6x$. In this case, the rate of change of the slopes of the tangent lines depends on *what point we are at.* The larger the magnitude of x, the more rapidly the curve 'turns'.



For example, when $x = 1$, we have $f''(1) = 6 \cdot 1 = 6$. Thus, when x changes by some small amount, we expect the slope of the tangent line to change by approximately six times this amount. Look at the chart below:



Observe that the change in f' is *approximately* six times the change in x, but not exactly. Why? The answer is quite simple: $f''(1)$ gives us an *instantaneous* rate of change. However, as soon as we move away from the point $(1, 1)$, the rate of change is no longer exactly 6. Indeed, over the interval $[1, 1.1]$, f'' is actually *greater* than 6; f'' increases from 6 (at $x = 1$) to 6.6 (at $x = 1.1$). This is precisely why our calculation was a bit high.

EXERCISE 5

Consider the function $f(x) = x^3$.

♣ 1. At the point $(2, 8)$, how fast are the slopes of the tangent lines changing?

♣ 2. How much do you estimate the slopes will change by, in moving from the point $(2, 8)$ to the point $(2.1, (2.1)^3)$?

♣ 3. Find the slopes of the tangent lines at both $x = 2$ and $x = 2.1$. What is $\Delta f'$?

♣ 4. Was your estimate high or low? Why?

EXERCISE 6

Let $f(x) = -(x - 4)^4 + 20$.

♣ 1. Sketch the graph of f.

♣ 2. Plot the point when $x = 2$.

♣ 3. How fast are the slopes of the the tangent lines changing when $x = 2$?

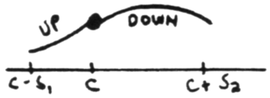
♣ 4. If we move to the point $(2.1, f(2.1))$, how much do you estimate the slopes will change by?

♣ 5. Find $f'(2)$ and $f'(2.1)$. How much did the slopes change by?

places where the concavity changes

Points on the graph of a function f where the concavity changes—from concave up to concave down, or from concave down to concave up—are particularly interesting. Thus, such points are given a special name—they are called *inflection points.* The precise definition appears next.

DEFINITION
inflection point



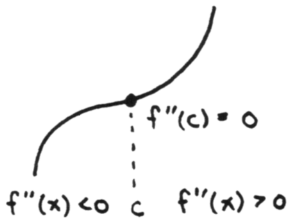
An *inflection point* is a point where the concavity of a function changes (from up to down, or down to up).

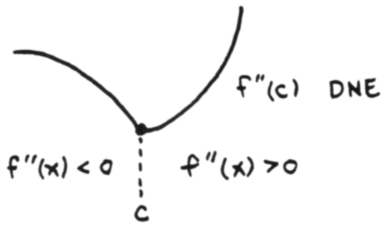
More precisely, the point $(c, f(c))$ is an *inflection point* for a function f if there is an interval $(c - \delta_1, c + \delta_2)$ about c, such that the concavity of f on $(c - \delta_1, c)$ differs from the concavity on $(c, c + \delta_2)$. Here, δ_1 and δ_2 are positive numbers.

Note that an inflection point cannot occur at an endpoint of the domain, because one has to be able to look on *both sides* to see if the concavity is different.

places where $f''(c) = 0$ or $f''(c)$ does not exist are the only CANDIDATES for inflection points

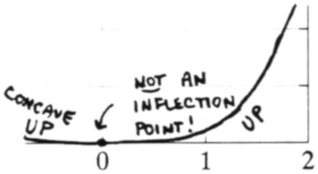
The sketches below illustrate two ways in which an inflection point can occur. It is possible to have an inflection point $(c, f(c))$ where $f''(c) = 0$. Also, it is possible to have an inflection point where $f''(c)$ does not exist. Indeed, a logical argument similar to that used in the previous section shows that *these are the only types of places where inflection points can occur.* Thus, the places where $f''(c) = 0$ and where $f''(c)$ does not exist give the *candidates* for places where inflection points occur.

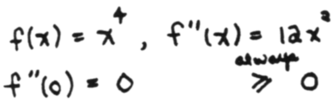




Caution!

If $f''(c) = 0$, this *does not mean* that there must be an inflection point at $(c, f(c))$. Similarly, if $f''(c)$ does not exist, this *does not mean* that there must be an inflection point at $(c, f(c))$. The sketches below illustrate this fact.





Recall that the critical points of a function give the *candidates* for places where local maxima and minima occur. Similarly, the places where $f''(c) = 0$ or $f''(c)$ does not exist merely give the *candidates* for the places where there are inflection points. Each of these points must be checked to see if it is, or is not, an inflection point.

strategy for finding the inflection points of a function

Suppose it is desired to find all the inflection points of a function f. Proceed as follows.

- Find the domain of f.
- Find f', and then f''. Find all $c \in \mathcal{D}(f)$ where $f''(c) = 0$ or $f''(c)$ does not exist. Remember that an inflection point cannot occur at an endpoint of the domain. These are the *candidates* for inflection points.
- Find the sign of f'' everywhere. Use this information to check each candidate.

EXAMPLE

finding inflection points

Problem: Find all inflection points for the function:

$$P(x) = x^4 + 4x^3 - 18x^2 - 6x + 1$$

- The domain of f is $\mathbb{R}$.
- Find P'':

$$\begin{aligned} P'(x) &= 4x^3 + 12x^2 - 36x - 6 \\ P''(x) &= 12x^2 + 24x - 36 \\ &= 12(x^2 + 2x - 3) \\ &= 12(x+3)(x-1) \end{aligned}$$

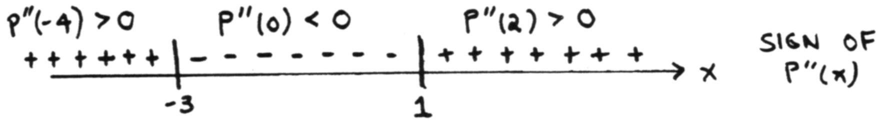
Observe that $\mathcal{D}(P'') = \mathbb{R}$, and:

$$P''(x) = 0 \iff x = -3 \text{ or } x = 1$$

When $x = 1$, $P(1) = (1)^4 + 4(1)^3 - 18(1)^2 - 6(1) + 1 = -18$. Thus, $(1, -18)$ is a candidate for an inflection point.

Similarly, $(-3, -170)$ is a candidate for an inflection point.

- Determine the sign of P'' everywhere:



To the left of $x = -3$, the graph is concave up; to the right, concave down. Thus, the concavity changes as one passes through the point $(-3, -170)$, so it is an inflection point. Similarly, $(1, -18)$ is an inflection point.

EXERCISE 7

Find all inflection points for each of the following functions:

♣ 1. $P(x) = x^4 - 4x^3 - 7x + 1$

♣ 2. $f(x) = \sqrt{x} + x^2$

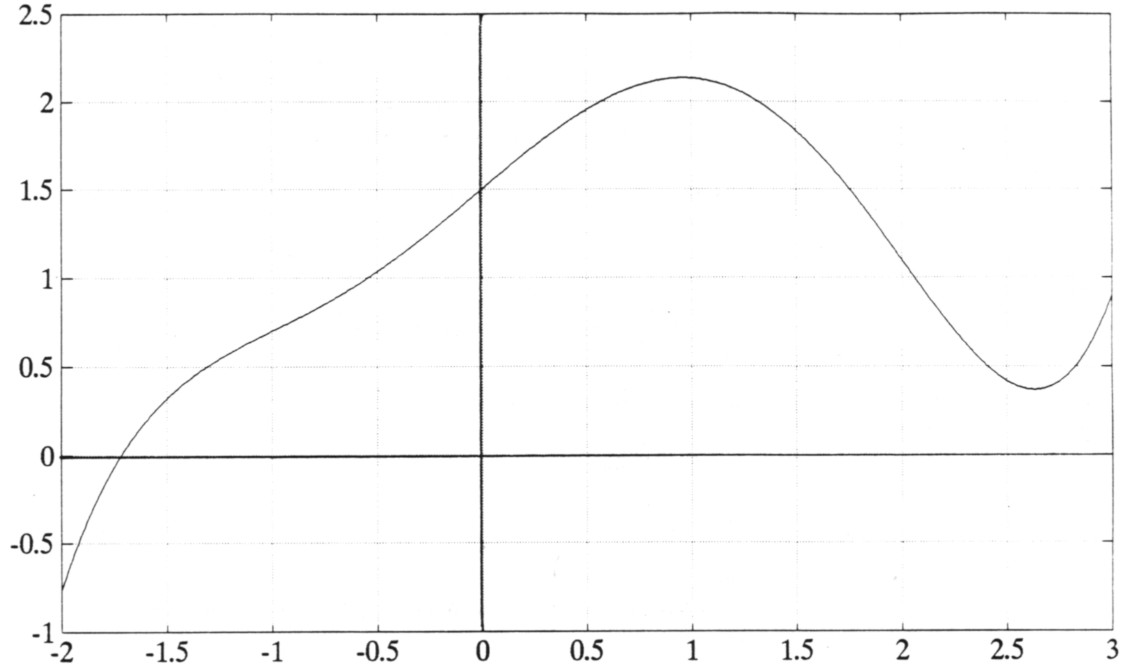
EXERCISE 8

Refer to the graph shown below to answer the following questions. Approximate where necessary.

♣ 1. On what open interval(s) is the function positive? Negative?

♣ 2. On what open interval(s) is the function increasing? Decreasing?

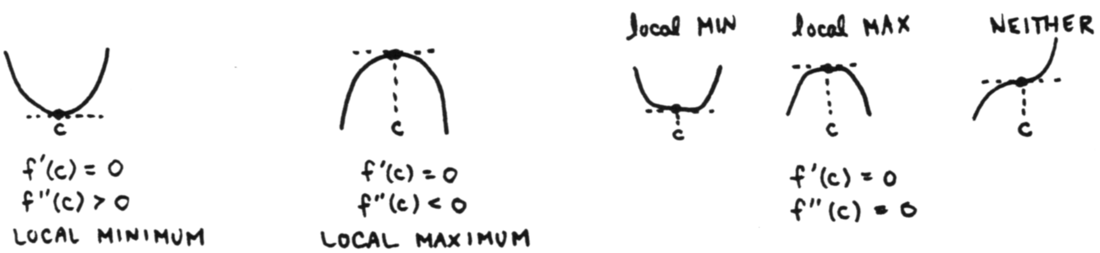
♣ 3. On what open interval(s) is the function concave up? Concave down?



a test for local extreme values that uses the second derivative

If $(c, f(c))$ is a critical point for f, and f is continuous at c, then one way to check if there is a local maximum or minimum at c is to investigate the sign of the first derivative near c. This is the content of the First Derivative Test.

In some cases, there is an easier test. Consider the sketches below. In each case, the point $(c, f(c))$ is a critical point because $f'(c) = 0$; that is, there is a horizontal tangent line at $(c, f(c))$.



In the first sketch, $f''(c) > 0$, so that the slopes of tangent lines are increasing as one passes through the point $(c, f(c))$. Since $f'(c) = 0$, it must be that the slopes are negative to the left of c, and positive to the right of c. That is, the function must decrease to the left of c, and increase to the right of c. Thus, $(c, f(c))$ must be a local minimum. That is, if $f''(c) > 0$, then the graph is concave up at c, and the point is a local minimum.

In the second sketch, $f''(c) < 0$. In this case, the graph is concave down at c, and the point is a local maximum.

If $f''(c) = 0$, anything is possible: no conclusion can be reached without further investigation. These observations lead to what is commonly known as the Second Derivative Test, stated and proved below.

The Second Derivative Test

for local maxima and minima

Suppose that $f'(c) = 0$, so that there is a horizontal tangent line at the point $(c, f(c))$.

If $f''(c) > 0$, then the point $(c, f(c))$ is a local minimum.

If $f''(c) < 0$, then the point $(c, f(c))$ is a local maximum.

If $f''(c) = 0$, no general conclusion is possible.

What does '$f''(c) > 0$' mean?

The proof is given for the case $f''(c) >$. The remaining case is left as an exercise.

One comment before we begin. When a mathematician says

$$f'(c) = 0 \ ,$$

this really means two things:

- f is differentiable at c, so that the number $f'(c)$ exists; and
- $f'(c) = 0$.

For the sake of brevity, the first sentence is usually omitted.

Similarly, when a mathematician says

$$f''(c) > 0 \ ,$$

this means that

- f is twice differentiable at c, so that the number $f''(c)$ exists; and
- $f''(c) > 0$.

PARTIAL PROOF

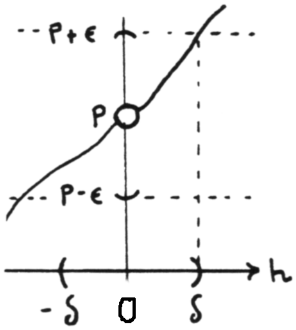
of the Second Derivative Test

Proof. Suppose that $f'(c) = 0$ and $f''(c) > 0$. Assume, for simplicity, that f is defined on both sides of c.

Recall that $f'' = (f')'$. Thus, $f''(c) > 0$ means that the limit

$$\lim_{h\to 0} \frac{f'(c+h) - f'(c)}{h} = \lim_{h\to 0} \frac{f'(c+h) - 0}{h}$$

exists, and is positive. Call the value of this limit P (for 'positive'). Thus, it is possible to get the values $\frac{f'(c+h)}{h}$ as close to P as desired, merely by requiring that h be sufficiently close to 0. Remember that when h is close to 0, $c+h$ is close to c. In particular, when $h < 0$, $c+h$ is to the left of c; and when $h > 0$, $c+h$ is to the right of c.



GRAPH OF $\frac{f'(c+h)}{h}$, FOR h NEAR 0

Refer to the sketch. Choose ϵ so that every number in the interval $I := (P - \epsilon, P + \epsilon)$ is positive. Then, find δ so that whenever h is within δ of 0, the numbers $\frac{f'(c+h)}{h}$ end up in I.

If $h < 0$, and within δ of 0, then multiplying both sides of the inequality

$$\frac{f'(c+h)}{h} > 0$$

by the negative number h yields

$$f'(c+h) < 0 ,$$

so the function is decreasing to the left of the point $(c, f(c))$.

Similarly, if $h > 0$ and within δ of 0, then we get

$$f'(c+h) > 0 ,$$

so the function is increasing to the right of the point $(c, f(c))$.

By the First Derivative Test, the point $(c, f(c))$ is a local minimum. ∎

EXERCISE 9

♣ 1. Prove the Second Derivative Test, in the case when $f''(c) < 0$.

♣ 2. Use the Second Derivative Test to find all local extreme values for $P(x) = 3x^4 + 4x^3 - 12x^2 + 1$. To do this, proceed as follows:

First, find all places where $P'(x) = 0$.

Next, check the sign of the second derivative at each value of c for which $P'(c) = 0$.

QUICK QUIZ

sample questions

1. What kind of information does the second derivative of a function give us?
2. Give a precise definition of what it means for a function f to be concave up on an interval I.
3. State the converse of this implication:

   If $x = 1$, then $x^2 = 1$

   Is the converse true or false?
4. Suppose that $f'(c) = 0$ and $f''(c) < 0$. What, if anything, can be said about the point $(c, f(c))$?
5. Let $f(x) = (x-1)^3$. Find $f''(1)$.

KEYWORDS *for this section*: *Concave up and down, the word 'conversely', inflection points, candidates for inflection points, strategy for finding inflection points, the Second Derivative Test.*

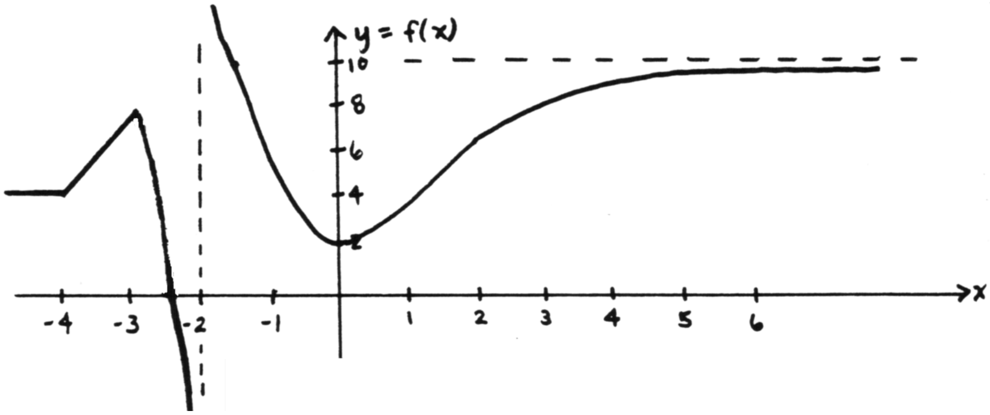
END-OF-SECTION EXERCISES

♣ Use BOTH the First and Second Derivative Tests to find all local extrema for the functions given below. Be sure that the results of both tests agree!

1. $P(x) = x^4 - 2x^3 + x^2 + 10$
2. $P(x) = 9x^4 + 16x^3 + 6x^2 + 1$

♣ All the remaining questions refer to the graph given below. Approximate where necessary. Assume that the patterns exhibited at the graph boundaries continue. If an object does not exist, so state.

♣ Read all the following information off the graph. Be sure to answer using complete mathematical sentences.



3. On what interval(s) is $f(x)$ positive? Negative?
4. On what interval(s) is f increasing? Decreasing?
5. On what interval(s) is f concave up? Concave down?
6. What is $\mathcal{D}(f)$?
7. What is $\mathcal{D}(f')$?
8. Find: $\{x \mid f'(x) = 0\}$
9. Find: $\{x \mid f(x) > 10\}$
10. Find: $\{x \mid f'(x) > 0\}$
11. Find: $\{x \mid f''(x) < 0\}$
12. Find: $\lim_{x\to 0} f(x)$
13. Find: $\lim_{t\to -2} f(t)$
14. Find: $\lim_{y\to -4} f(y)$
15. List all the critical points for this function.
16. Find: $f(0)$, $f'(0)$, $f(1000)$, $f'(1000)$
17. Find: $\{x \in \mathcal{D}(f) \mid f \text{ is not differentiable at } x\}$
18. Find all inflection points.
19. Find: $\displaystyle\lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$
20. Find: $\displaystyle\lim_{x\to -3.5} f'(x)$