6.1 Antiderivatives

undoing differentiation

In the previous sections, the focus has been on differentiating a given function: given f, find f'.

The question for the current chapter is this: given a function f, find another function F *whose derivative is* f. That is, given f, we seek F such that $F' = f$. So, in a sense, we are *undoing* differentiation.

a preliminary example

Let $f(x) = 3$. We want another function whose derivative is f. That is, we seek a function F satisfying $F'(x) = f(x) = 3$.

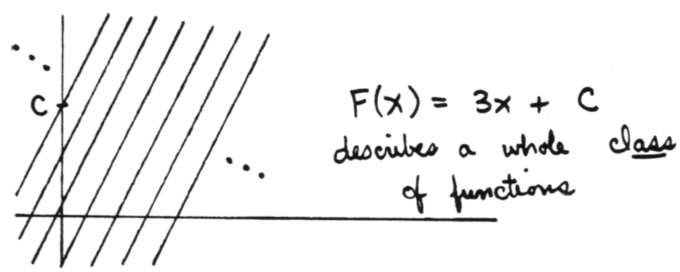
Clearly, $F(x) = 3x$ works, since in this case $F'(x) = 3$.

Also, $F(x) = 3x + 1$ works, since again $F'(x) = 3$.

Indeed, for any real number C, $F(x) = 3x + C$ is a function whose derivative is 3.

the equation $F(x) = 3x + C$ describes an ENTIRE CLASS of functions

Observe that the equation $F(x) = 3x + C$ describes an *entire class* of functions which have the *same shape*, but are translated up and down in the xy-plane. There is one function for each choice of the number C.



What does ANY function with derivative 3 look like?

Are there any functions other than those of the form $F(x) = 3x + C$ whose derivative is 3? We will see momentarily that the answer is 'No'.

an application of the Mean Value Theorem

Here's the way mathematicians address such a question: they *suppose* there is a function with derivative 3, and then proceed to show that it must actually be of the form $3x + C$.

Whenever derivative information is to be used to glean information about the function itself, you should not be surprised to see the Mean Value Theorem. Make sure you see how the Mean Value Theorem plays a crucial role in the next argument.

Suppose G is ANY function with derivative 3 ...

Let $F(x) = 3x+C$, where C is any real number. Suppose that G is any function with derivative 3. Observe that we are *not* assuming that G must be of the form $G(x) = 3x + C$.

Then, we have both
$$F'(x) = 3 \text{ and } G'(x) = 3\ ,$$
so that:
$$G'(x) - F'(x) = 0$$

Since the sum of the derivatives is the derivative of the sum, we can alternately write:

$$(G - F)'(x) = 0$$

Now recall a result from the end of Chapter 4. There, we learned that if the derivative of a function is zero, then the function must be constant: this was an application of the Mean Value Theorem. Thus, we must have

$$(G - F)(x) = K$$

for some constant K. That is,

$$G(x) - F(x) = K \ ,$$

or:

$$G(x) = F(x) + K = (3x + C) + K$$

... then, G must be of the form $3x + C$

Thus, we see that G *must actually be* of the form $3x +$ (some constant). It has therefore been established that *every* function with derivative 3 must look like $3x + C$ for some constant C.

the derivative of a function completely determines its shape

The preceding argument is now generalized slightly. Suppose that functions f and g are both differentiable (say on an open interval (a, b)), and suppose that:

$$f'(x) = g'(x) \ \ \forall \ x \in (a, b)$$

Then,

$$(f - g)'(x) = f'(x) - g'(x) = 0 \ ,$$

so that $(f - g)(x) = C$ for some constant C. That is,

$$f(x) - g(x) = C \ ,$$

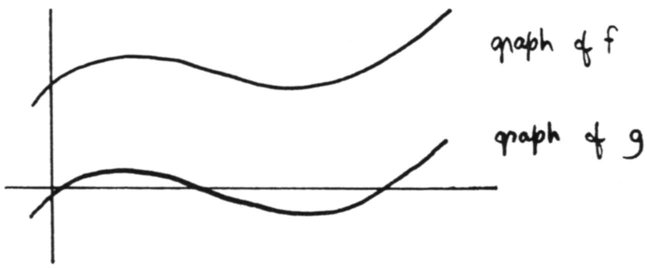
and hence:

$$f(x) = g(x) + C$$

Thus, *if two functions f and g have the same derivative, then they differ by at most a constant.* That is, *functions that have the same derivative must have the same shape.* The functions f and g might not be the same function, but the graph of one can be obtained from the graph of the other by a vertical translation.

In other words, *specifying the derivative of a function completely determines its shape.*

f and g have the same derivative



EXERCISE 1	♣ Let $f(x) = -1$. Find all functions F for which $F' = f$. How many are there? Sketch a few such functions F.

EXERCISE 2

Consider the function f with graph shown below. Note that $\mathcal{D}(f) = \mathbb{R} - \{c\}$.

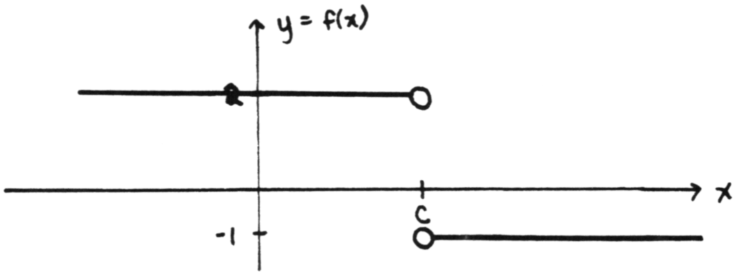
Sketch the graph of a function F satisfying each of the following properties:

♣ 1. F is continuous on $\mathbb{R}$, and $F'(x) = f(x)$ for all $x \in \mathcal{D}(f)$

♣ 2. Sketch another, different, function F satisfying the requirements above.

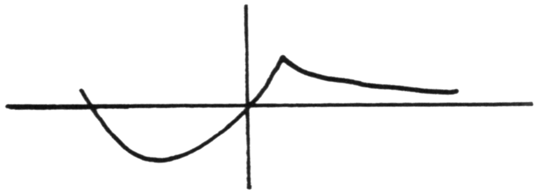
♣ 3. $\mathcal{D}(F) = \mathcal{D}(f)$, $F'(x) = f(x)$ for all $x \in \mathcal{D}(f)$, and F has a removable discontinuity at c

♣ 4. $\mathcal{D}(F) = \mathcal{D}(f)$, $F'(x) = f(x)$ for all $x \in \mathcal{D}(f)$, and F has a nonremovable discontinuity at c



EXERCISE 3

♣ Consider the function f shown below. On the same graph, sketch two different functions that have the same derivative as f.



DEFINITION
antiderivative;
arbitrary constant

A function F is called an *antiderivative* of a function f if

$$F'(x) = f(x)$$

for every x in the domain of f.

Thus, an antiderivative of f is a function *whose derivative is* f.

If you are able to find a *single* antiderivative of f, call it F, then there are an *infinite* number of antiderivatives, each of the form:

$$F(x) + C$$

Here, C represents any real number, and is called an *arbitrary constant.*

NOTATION
for antiderivatives:
indefinite integrals;
antidifferentiation;
integral sign;
integrand

The symbol

$$\int f(x)\,dx$$

is called the *indefinite integral of* f, and represents *all the antiderivatives of* f.

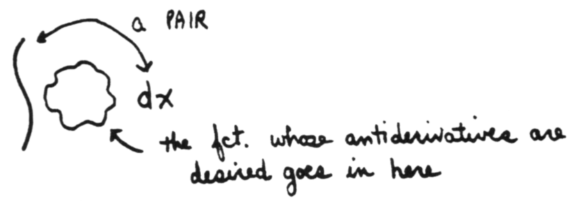
The process of finding $\int f(x)\,dx$ is called *antidifferentiation* (“undoing” differentiation!)

The symbol $\int$ is called the *integral sign.*

The function f that is being integrated is called the *integrand.*

$\int$ and dx are an instruction pair

It may be helpful to view the integral sign $\int$ and the symbol dx as an *inseparable instruction pair*. The function of x (call it f) whose antiderivatives are desired is placed between the symbols $\int$ and dx. The instruction is then to find all functions, whose derivatives *with respect to* x, equal f.



more notation: integrals; integration

Later on, we will study the *definite integral of f on* $[a, b]$, to be denoted by the symbol $\int_a^b f(x)\,dx$.

Both $\int f(x)\,dx$ (the indefinite integral) and $\int_a^b f(x)\,dx$ (the definite integral) are called *integrals.*

The process of finding either $\int f(x)\,dx$ or $\int_a^b f(x)\,dx$ is called *integration.*

EXAMPLE

the constant C is often called the 'constant of integration'

Problem: Evaluate $\int 3\,dx$.

Solution: We are asked to find all the antiderivatives of the function $f(x) = 3$. That is, we are asked to find all functions of x, whose derivative with respect to x is 3 . The solution is written concisely and correctly as:

$$\int 3\,dx = 3x + C$$

It is conventional that the letters C or K be used in this context to represent an arbitrary constant (i.e., any real number). This arbitrary constant is also referred to as the *constant of integration.*

It is important that you include the constant of integration. If you mistakenly write

$$\int 3\,dx = 3x \ ,$$

then you are claiming that the ONLY function whose derivative is 3 is the function $3x$. Not so! This is only *one* of an *infinite class* of functions that has derivative 3!

checking your answers by differentiating

If $\int f(x)\,dx = F(x) + C$, then $F(x) + C$ is an antiderivative of $f(x)$, so that:

$$\frac{d}{dx}\big(F(x) + C\big) = F'(x) + 0 = f(x)$$

Thus, answers can always be checked by differentiating.

Since the derivative of a constant is always zero, it is not necessary to include 'C' in the checking process. More simply, check that:

$$\frac{d}{dx}F(x) = F'(x) = f(x)$$

For example, to check that

$$\int 3\,dx = 3x + C\ ,$$

CHECK!

one verifies that:

$$\frac{d}{dx}(3x) = 3$$

practice with notation

Observe what happens when the element dx is changed:

$$\int 3\,dx = 3x + C$$

$$\int 3\,dy = 3y + C$$

$$\int 3\,d\omega = 3\omega + C$$

In the first case, $\frac{d}{dx}(3x) = 3$.

In the second case, $\frac{d}{dy}(3y) = 3$.

In the third case, $\frac{d}{d\omega}(3\omega) = 3$.

EXAMPLE

Problem: Evaluate $\int 2x\,dx$.

Solution: It is necessary to find *any* antiderivative of $2x$; that is, a function with derivative $2x$. Then, all other antiderivatives will differ by at most a constant.

Observe that $F(x) = x^2$ works, since $F'(x) = 2x$.

Once we have a *single* antiderivative, we actually know them all. That is, any other function with the same derivative must have precisely the same shape. So:

$$\int 2x\,dx = x^2 + C$$

CAUTION!

Be careful not to write something like this:

$$\int 2t\,dx = t^2 + C$$

Taken literally, this says that

$$\frac{d}{dx}t^2 = 2t\ ,$$

which doesn't make any sense: if we're differentiating with respect to x, and are not told otherwise, then we would have to assume that t is constant with respect to x. Thus, $\frac{d}{dx}t^2 = 0$. The MORAL: make sure the letter x in the element 'dx' agrees with the variable of the function that you're integrating! (Unless, of course, you're doing something unusual.)

EXERCISE 4

Evaluate the following indefinite integrals. Be sure to write complete sentences. Don't forget to include the constant of integration.

♣ 1. $\int 3x^2\,dx$

♣ 2. $\int 2y\,dy$

♣ 3. $\int e^t\,dt$

♣ 4. $\int 2e^{2x}\,dx$

♣ 5. $\int \frac{1}{x}\,dx$ Here, just find an antiderivative of $\frac{1}{x}$ on the interval $(0,\infty)$.

EXERCISE 5

For this exercise, assume that $x > 0$, so that $\ln x$ is defined.

Recall that $F(x) = \ln x$ has derivative $F'(x) = \frac{1}{x}$. Thus, $\ln x$ is an antiderivative of $\frac{1}{x}$, and hence:

$$\int \frac{1}{x}\,dx = \ln x + C \tag{*}$$

Also, $G(x) = \ln 2x$ has derivative $G'(x) = \frac{1}{2x}(2) = \frac{1}{x}$. Thus, $\ln 2x$ is an antiderivative of $\frac{1}{x}$, and hence:

$$\int \frac{1}{x}\,dx = \ln 2x + K \tag{**}$$

Equation (*) tells us that *every* antiderivative of $\frac{1}{x}$ must be of the form $\ln x + C$ for some constant C.

Equation (**) tells us that *every* antiderivative of $\frac{1}{x}$ must be of the form $\ln 2x + K$ for some constant K.

♣ Reconcile these two statements. That is, how can they both be true?

linearity of differentiation

If f and g are both differentiable functions of x, then

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

and:

$$\frac{d}{dx}k \cdot f(x) = k \cdot f'(x)$$

That is, the derivative of a sum is the sum of the derivatives, and constants can be 'slid out' of the differentiation process.

These two properties together are referred to as the *linearity of differentiation*. Alternately, one often says '*differentiation is a linear process*'.

We see next that the process of *antidifferentiation* obeys the same two properties: the integral of a sum is the sum of the integrals, and constants can be 'slid out' of the integral.

EXERCISE 6

♣ 1. What is meant by the phrase '*linearity of differentiation*'?

♣ 2. Identify all the places where the linearity of differentiation is used in the following sentence:

$$\begin{aligned}\frac{d}{dx}(x^2+3\sqrt{x}) &= \frac{d}{dx}(x^2+3x^{1/2}) \\ &= \frac{d}{dx}x^2 + \frac{d}{dx}3x^{1/2} \\ &= 2x + 3\frac{d}{dx}x^{1/2} \\ &= 2x + 3(\frac{1}{2}x^{-1/2}) \\ &= 2x + \frac{3}{2\sqrt{x}}\end{aligned}$$

linearity of integration

Integration is a linear process, as is differentiation.

That is, the integral of a sum is the sum of the integrals:

$$\int f(x)+g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$$

Also, constants can be slid out of the integration process:

$$\int k\,f(x)\,dx = k\int f(x)\,dx$$

Together, these two properties are referred to as the *linearity of the integral* or the *linearity of integration.*

partial proof of the linearity of integration

The fact that antidifferentiation is a linear process is a direct consequence of the linearity of differentiation, as the following discussion illustrates.

Problem: Show that:

$$\int \big(f(x)+g(x)\big)\,dx = \int f(x)\,dx + \int g(x)\,dx \qquad (\dagger)$$

Solution: To begin, let F be an antiderivative of f (so that $F' = f$) and let G be an antiderivative of g (so that $G' = g$).

Then,

$$\int f(x)\,dx = F(x) + C_1$$

and

$$\int g(x)\,dx = G(x) + C_2\ ,$$

where C_1 and C_2 are arbitrary constants.

Then, the right-hand side of (†) becomes:

$$\begin{aligned}\int f(x)\,dx + \int g(x)\,dx &= (F(x)+C_1) + (G(x)+C_2)\\ &= F(x)+G(x)+(C_1+C_2)\\ &= F(x)+G(x)+C \qquad (1)\end{aligned}$$

Here, the two arbitrary constants have been lumped together into a single arbitrary constant.

Next, investigate the left-hand side of (†). What is $\int \big(f(x)+g(x)\big)\,dx$? We need a function with derivative $f(x)+g(x)$. But $F(x)+G(x)$ is such a function:

$$\frac{d}{dx}(F(x)+G(x)) = F'(x)+G'(x) = f(x)+g(x)$$

Thus, the left-hand side of (†) becomes:

$$\int \big(f(x)+g(x)\big)\,dx = F(x)+G(x)+C \qquad (2)$$

Compare (1) and (2)—they are identical. Thus, it has been shown that

$$\int \big(f(x)+g(x)\big)\,dx = \int f(x)\,dx + \int g(x)\,dx\ ,$$

establishing that the integral of a sum is the sum of the integrals.

EXERCISE 7

♣ 1. Similar to the preceding argument, prove that:

$$\int kf(x)\,dx = k\int f(x)\,dx$$

That is, constants can be 'slid out' of the integral.

♣ 2. Is $\int x^2\,dx = x\int x\,dx$? That is, can an '$x$' be slid out of the integral? Comment.

EXAMPLE

using the linearity of integration

The linearity of the integral can be used to solve a wide variety of integration problems. For example:

$$\begin{aligned}\int (2x-3)\,dx &= \int 2x\,dx + \int (-3)\,dx\\ &= \int 2x\,dx - \int 3\,dx\\ &= (x^2+C_1)-(3x+C_2)\\ &= x^2-3x+(C_1-C_2)\\ &= x^2-3x+C\end{aligned}$$

All arbitrary constants are always lumped into a single arbitrary constant. The previous problem is never written out in all the detail shown above. It is more simply written as:

$$\int (2x-3)\,dx = x^2 - 3x + C$$

That is: find an antiderivative of $2x$, subtract an antiderivative of 3, and add on an arbitrary constant.

EXERCISE 8 ♣ Supply a reason for *each line* in this mathematical sentence:

$$\begin{aligned}\int (2x-3)\,dx &= \int 2x\,dx + \int (-3)\,dx\\ &= \int 2x\,dx - \int 3\,dx\\ &= (x^2 + C_1) - (3x + C_2)\\ &= x^2 - 3x + (C_1 - C_2)\\ &= x^2 - 3x + C\end{aligned}$$

EXAMPLE Often, it is necessary to rewrite the integrand before integrating:

$$\begin{aligned}\int \frac{e^x - 1}{e^x}\,dx &= \int \frac{e^x}{e^x} - \frac{1}{e^x}\ dx\\ &= \int 1 - e^{-x}\ dx\\ &= x + e^{-x} + C\end{aligned}$$

Check:

$$\begin{aligned}\frac{d}{dx}(x + e^{-x}) &= 1 - e^{-x}\\ &= \frac{e^x}{e^x}(1 - e^{-x})\\ &= \frac{e^x - 1}{e^x}\end{aligned}$$

EXAMPLE As a second example:

$$\begin{aligned}\int \frac{2}{3x-7}\,dx &= 2\int \frac{1}{3(x-\frac{7}{3})}\,dx\\ &= \frac{2}{3}\int \frac{1}{x-\frac{7}{3}}\,dx\\ &= \frac{2}{3}\ln(x - \frac{7}{3}) + C\end{aligned}$$

♣ Do you see where two arbitrary constants were combined in this argument?

Check:

$$\begin{aligned}\frac{d}{dx}\frac{2}{3}(\ln(x-\frac{7}{3}) &= \frac{2}{3}(\frac{1}{x-\frac{7}{3}})\\ &= \frac{2}{3x-7}\end{aligned}$$

♣ Do you see where the linearity of differentiation was used in this check?

In the next few sections, additional tools are developed to help in the integration process.

EXERCISE 9

Evaluate the following integrals. Be sure to write complete mathematical sentences. Don't forget to include the constant of integration.

♣ 1. $\int(\frac{1}{x}+e^x-1)\,dx$

♣ 2. $\int\frac{3-t}{t}\,dt$

♣ 3. $\int\frac{1}{x-2}\,dx$

♣ 4. $\int\frac{1}{3x-5}\,dx$

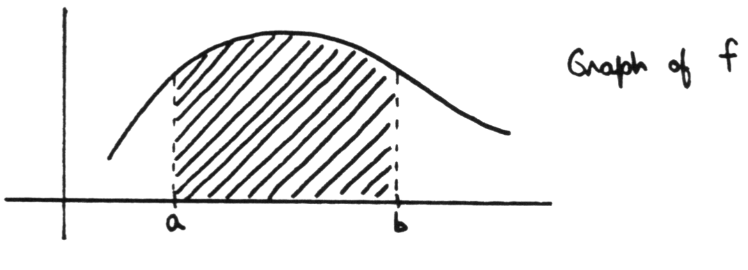
♣ 5. $\int(x+1)^2\,dx$

a preview of coming attractions

We will soon learn a very surprising fact: if f is a continuous nonnegative function, and if we can find an antiderivative F of f, then we can use this antiderivative to find the area trapped between the graph of f and the x-axis over an interval $[a,b]$!

All we have to do is this: evaluate the antiderivative F at b to get $F(b)$. Evaluate F at a, to get $F(a)$. Then:

$$\text{desired area} = F(b)-F(a)$$



This result is properly discussed in the next chapter. For now, just keep in mind that the antiderivatives of a function have a *very practical use*! To close this section, we look at a simple example of this surprising connection between antiderivatives and area.

EXAMPLE

finding area using an antiderivative

Problem: Find the area trapped beneath the graph of $f(x) = 2x$ on the interval $[a, b]$, where $0 < a < b$.

Solution: The desired area is a trapezoid, and calculus is certainly *not* needed, in this case, to find it:

$$\begin{aligned} \text{desired area} &= \frac{1}{2}(\text{altitude})(\text{sum of bases}) \\ &= \frac{1}{2}(b-a)(2a+2b) \\ &= \frac{1}{2}(b-a)2(a+b) \\ &= (b-a)(b+a) \end{aligned}$$

Now, let's use calculus to get the same answer. This time, we first find an antiderivative of f:

$$F(x) = x^2 \quad \text{has derivative} \quad F'(x) = 2x = f(x)$$

Then:

$$F(b) - F(a) = b^2 - a^2 = (b-a)(b+a)$$

Note that precisely the same result is obtained!

EXERCISE 10

♣ 1. Graph $f(x) = x$.

♣ 2. On your graph, show the area trapped beneath the graph of f and the x-axis on an interval $[a, b]$, where $0 < a < b$.

♣ 3. Compute this area, using the formula for the area of a trapezoid.

♣ 4. Next, observe that $F(x) = \frac{x^2}{2}$ is an antiderivative of f, since $F'(x) = \frac{1}{2}(2x) = x = f(x)$. Use calculus to find the area being investigated. Compare your answers.

QUICK QUIZ

sample questions

1. Suppose a function $f(x)$ has derivative 2 everywhere. What does the graph of f look like?
2. Fill in the Blank: specifying the derivative of a function completely determines its ________________.
3. Find $\int 2\,dt$.
4. Name one use for the antiderivatives of a function.
5. What is meant by the phrase, 'the linearity of differentiation'?

KEYWORDS

for this section

'Undoing' differentiation, the derivative of a function completely determines its shape, antiderivative, arbitrary constant, indefinite integrals, antidifferentiation, integral sign, integrand, integrals, integration, constant of integration, linearity of differentiation, linearity of integration, connection between antiderivatives and area.

END-OF-SECTION EXERCISES

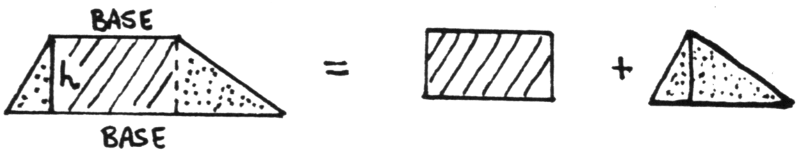
♣ Classify each entry below as an expression or a sentence.

♣ For any *sentence*, state whether it is TRUE, FALSE, or CONDITIONAL.

(Feel free to assume that all functions appearing below are infinitely differentiable.)

1. $F'(x)$
2. $F'(x) = 2$
3. $\int f(x)\,dx$
4. $\int f(t)\,dt$
5. $\int f(x)\,dx = F(x) + C$
6. $\int 2\,dx = 2x + C$
7. $\int 2\,dt = 2t + C$
8. $\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$
9. $\int kf(x)\,dx = k \int f(x)\,dx$
10. $\int f'(x)\,dx = f(x) + C$
11. (Deriving the formula for the area of a trapezoid) A trapezoid is any quadilateral with two parallel sides. The distance between the two parallel sides is called the *altitude* of the trapezoid. The two parallel sides are called the *bases* of the trapezoid.

    The area of any trapezoid can be found as the sum of a rectangle and a triangle, as illustrated below:



♣ Sum the areas of the rectangle and triangle, and conclude that:

$$\text{area of a trapezoid} = \frac{1}{2}(\text{altitude})(\text{sum of bases})$$