6.3 Analyzing a Falling Object
(Optional)

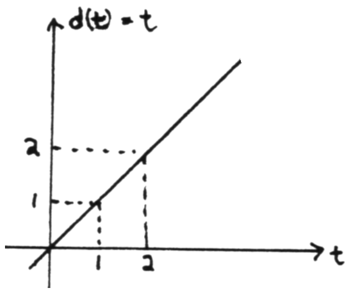
Introduction

In this section the motion of a falling object that is acted upon only by gravity is studied; this is a beautiful application of antidifferentiation to a real-life problem. Such an object travels in a (vertical) line, and it is thus first necessary to understand motion along a line. This is the next topic of discussion.

a particle traveling along a line

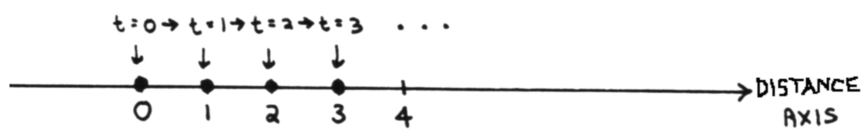
Suppose the function d tells the position of a particle along a line at time t. For convenience, distance along the line is measured in units of feet; time is measured in seconds.

For example, the function $d(t) = t$ describes a particle that is:

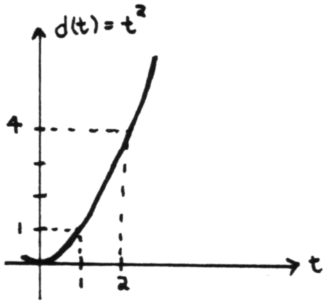


$$
\begin{array}{llll}
\text{at position} & d(0) = 0 & \text{at} & t = 0 \\
\text{at position} & d(1) = 1 & \text{at} & t = 1 \\
\text{at position} & d(2) = 2 & \text{at} & t = 2 \\
\text{at position} & d(T) = T & \text{at} & t = T
\end{array}
$$

The particle travels to the right at a constant speed of 1 foot per second.

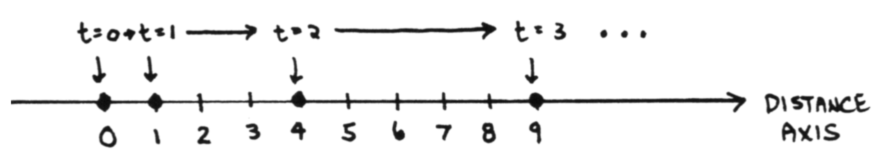


The function $d(t) = t^2$ describes a particle that is:

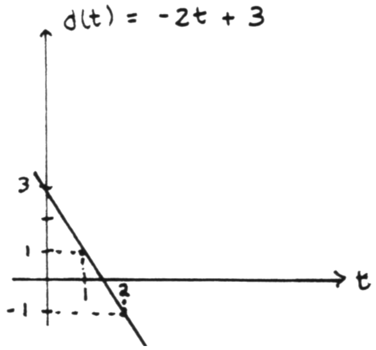


$$
\begin{array}{llll}
\text{at position} & d(0) = 0 & \text{at} & t = 0 \\
\text{at position} & d(1) = 1 & \text{at} & t = 1 \\
\text{at position} & d(2) = 4 & \text{at} & t = 2 \\
\text{at position} & d(T) = T^2 & \text{at} & t = T
\end{array}
$$

The particle travels to the right, and continually picks up speed as it travels.

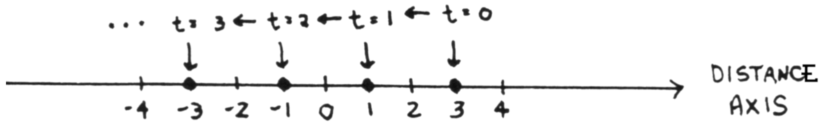


The function $d(t) = -2t + 3$ describes a particle that is:



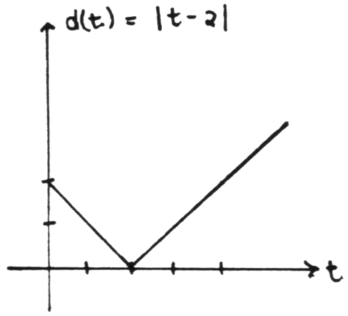
$$
\begin{aligned}
&\text{at position } \ d(0) = 3 \ \text{ at } \ t = 0 \\
&\text{at position } \ d(1) = 1 \ \text{ at } \ t = 1 \\
&\text{at position } \ d(2) = -1 \ \text{ at } \ t = 2 \\
&\text{at position } \ d(3) = -3 \ \text{ at } \ t = 3 \\
&\text{at position } \ d(T) = -2T + 3 \ \text{ at } \ t = T
\end{aligned}
$$

This particle starts at position 3, and travels to the left at a uniform speed of 2 feet per second.

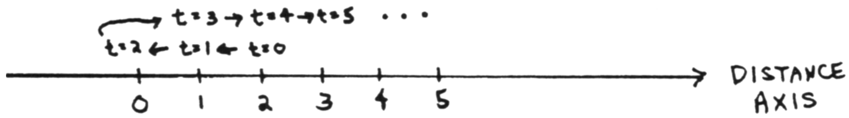


The function $d(t) = |t - 2|$ describes a particle that is:

$$
\begin{aligned}
&\text{at } \ d(0) = 2 \ \text{ at } \ t = 0 \\
&\text{at } \ d(1) = 1 \ \text{ at } \ t = 1 \\
&\text{at } \ d(2) = 0 \ \text{ at } \ t = 2 \\
&\text{at } \ d(3) = 1 \ \text{ at } \ t = 3 \\
&\text{at } \ d(4) = 2 \ \text{ at } \ t = 4 \\
&\text{at } \ d(T) = T - 2 \ \text{ at } \ T > 2
\end{aligned}
$$



This particle starts at 2, moves backward to zero, then turns around and travels to the right. Except when it turns, the particle moves at a constant speed of one unit per second.



EXERCISE 1 A particle traveling along a line is at position $d(t)$ feet at t seconds. Describe the resulting motion, if:

♣ 1. $d(t) = 3t$

♣ 2. $d(t) = -3t$

♣ 3. $d(t) = -t^2$

♣ 4. $d(t) = 2|t - 1|$

instantaneous velocity, $v(t) := d'(t)$

Recall the instantaneous rate of change interpretation of the derivative: $f'(c)$ gives the instantaneous rate of change of the numbers $f(x)$ with respect to x, at the point $(c, f(c))$.

Specializing to the current setting, let $d(t)$ represent the position of a particle at time t. Then, $d'(T)$ gives the instantaneous rate of change of the numbers $d(t)$ with respect to time t, at the point $(T, d(T))$. That is, $d'(T)$ gives a *change in distance* per *change in time*. This type of information is commonly called *velocity*. Thus, $d'(t)$ gives the (instantaneous) velocity at time t, and is commonly denoted by $v(t)$. That is:

$$v(t) := d'(t) = \text{instantaneous velocity at time } t$$

Remember that ':=' means '*equals, by definition*'. Here, the name $v(t)$ ('v', for velocity) is being *assigned* to the derivative $d'(t)$. If distance along the line is measured in units of feet, and time is measured in seconds, then:

$$\text{units of } v(t) = \frac{\text{units of position}}{\text{units of time}} = \frac{\text{feet}}{\text{second}}$$

EXAMPLE

finding $v(t)$

Consider the earlier examples.

When $d(t) = t$, then $v(t) := d'(t) = 1$. At *every* time t, the instantaneous velocity is 1 foot per second. No matter where the particle is currently sitting on the line, it travels to the right at one foot per second.

EXAMPLE

When $d(t) = t^2$, then $v(t) := d'(t) = 2t$. In this case, the velocity of the particle *depends on* the time at which we are investigating the particle.

At $t = 0$, the particle is at position $d(0) = 0^2 = 0$ ft, and has instantaneous velocity $d'(0) = 2 \cdot 0 = 0$ ft/sec.

At $t = 1$, the particle is at position $d(1) = 1^2 = 1$ ft, and has instantaneous velocity $d'(1) = 2 \cdot 1 = 2$ ft/sec.

At $t = 2$, the particle is at position $d(2) = 2^2 = 4$ ft, and has instantaneous velocity $d'(2) = 2 \cdot 2 = 4$ ft/sec.

At $t = 3$, the particle is at position $d(3) = 3^2 = 9$ ft, and has instantaneous velocity $d'(3) = 2 \cdot 3 = 6$ ft/sec.

The particle moves faster and faster as it travels along the line.

EXERCISE 2

♣ Find $v(t)$ for each of the distance functions from Exercise 1. Does this velocity information agree with the description of the motion you gave in Exercise 1?

EXAMPLE

'velocity' versus 'speed'

When $d(t) = -2t + 3$, then $v(t) := d'(t) = -2$. At *every* time t, the particle has velocity -2 ft/second. That is, when t increases by 1, $d(t)$ *decreases* by 2. Thus, the negative sign indicates that the particle is moving to the left.

The word *speed* is commonly used to describe how fast something moves, regardless of the *direction* in which it moves. For example, if a particle travels to the right, covering 2 feet per second, it has speed 2 ft/second. If a particle travels to the left, covering 2 feet per second, it still has speed 2 ft/second.

Precisely, the *speed* of a particle at time t is given by the *magnitude* of velocity. That is:

$$\text{speed at time } t = |v(t)|$$

Observe that *velocity has both magnitude (size) and direction*, but *speed has only magnitude*.

EXAMPLE

Problem: Suppose the position of a particle traveling along a line is given by $d(t) = t^2 - 5t + 3$. Find the position, velocity, and speed of the particle at $t = 1$. Suppose distance along the line is measured in meters; time is measured in minutes.

Solution: The position of the particle at $t = 1$ is $d(1) = 1^2 - 5 \cdot 1 + 3 = -1$ meters.

$v(t) := d'(t) = 2t - 5$; so the velocity at $t = 1$ is $v(1) = 2 \cdot 1 - 5 = -3$ meters/minute.

The speed at $t = 1$ is $|v(1)| = |-3| = 3$ meters/minute.

At $t = 1$, the particle is traveling to the left, at the rate of 3 meters per minute.

EXERCISE 3

♣ Suppose the position of a particle traveling along a line is given by $d(t) = t^3 - 2t^2 + 3$. Suppose distance is measured in meters, and time is measured in seconds. Find the position, velocity, and speed of the particle at: $t = 1$, $t = -1$, $t = 0$, $t = T$

instantaneous acceleration, $a(t) := v'(t) = d''(t)$

A change in velocity per change in time is commonly called *acceleration*. For example, when a car 'accelerates', this means that its speed is increasing.

The function v' gives the change in velocity per change in time. Thus, this function v' is renamed a, and called the 'acceleration function'. Observe that $v'(t) = \frac{d}{dt}v(t) = \frac{d}{dt}d'(t) = d''(t)$. Precisely:

$$a(t) := v'(t) = d''(t) = \text{instantaneous acceleration at time } t$$

What are the units of acceleration? Since acceleration is a change in velocity per change in time, it has units of $\frac{\text{velocity}}{\text{time}}$. For example, if distance is measured in feet and time in seconds, then:

$$\text{units of acceleration} = \frac{\text{ft/sec}}{\text{sec}} = \frac{\text{ft}}{\text{sec}^2}$$

Going 'backwards': when you see units of (say) ft/sec^2, it may be valuable to remind yourself that this is 'feet per second, per second'.

For example, consider the distance function $d(t) = t$. Here, differentiating once yields $v(t) = 1$, and differentiating once more yields $a(t) = 0$. The particle always travels to the right with speed 1. Its velocity is not changing. Thus, its acceleration is 0.

Next consider the distance function $d(t) = t^2$. Here, $v(t) = 2t$ and $a(t) = 2$. When time increases by 1, the velocity of the particle increases by 2. The particle is speeding up. And no matter what time we look at the particle, it is always speeding up at the same rate. It has a constant acceleration of 2 ft/sec^2.

EXERCISE 4

♣ 1. Find the acceleration functions for each of the distance functions from Exercise 1. Think about your results.

♣ 2. Find the acceleration function for $d(t) = 2t^3 + t^2 - 3t + 1$.

vectors

A *vector* is a mathematical object that is completely characterized by two pieces of information: a magnitude (size, absolute value) and a direction. Vectors are conveniently represented using arrows: the length of the arrow represents the magnitude of the vector; the direction that the arrow is pointing represents the direction of the vector. The directions that vectors are allowed to take on is determined by the 'space' in which the vectors live, as illustrated by the examples below.

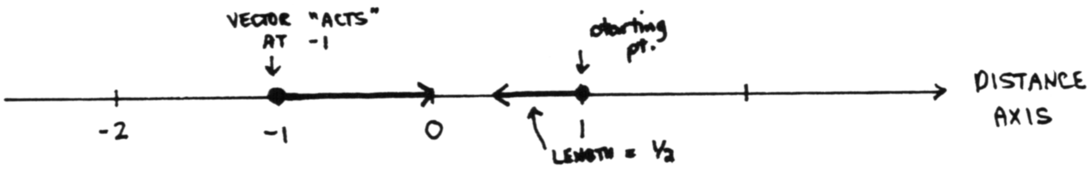
vectors in a line

Suppose the 'space' in which the vectors 'live' is a line. In a line, there are only two possible directions to move. If the line is positioned so that it is horizontal, these two directions are conveniently referred to as 'left' and 'right'. If the line is positioned so that it is vertical, these two directions are conveniently referred to as 'up' and 'down'. For other orientations of the line, names for the two directions are not so clear.

the starting point of a vector shows where it is 'acting'

Some vectors in a line are shown below. Note that each vector has a starting point (the non-arrow end). This starting point indicates where the vector is 'acting'.

For example, if vectors are being used to display velocity information of a particle traveling along a line (distance measured in feet, time in seconds) then the right-most vector below shows that when the particle is at position 1, it is moving left at a speed of $\frac{1}{2}$ foot/sec. The left-most vector below shows that when the particle is at position -1, it is moving right at a speed of 1 foot/sec.



EXERCISE 5

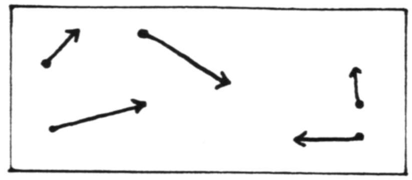
♣ 1. Suppose that when a particle is at position 5 on a line, it is moving left at 2 feet/sec. Illustrate this information using a vector.

♣ 2. Suppose that when a particle is at position -2 on a line, its velocity is 1 ft/sec. Illustrate this information using a vector.

♣ 3. The distance function $d(t) = t^2 - 1$ describes a particle's motion along a line (distance in feet, time in seconds). Illustrate the velocity information on a distance axis, at $t = 2$.

vectors in the plane

If vectors 'live' in a plane, then there are a lot more directions to move. Some vectors in a plane are illustrated below.



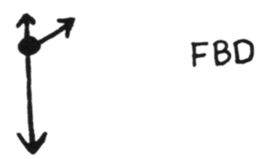
free-body diagram

We are now in a position to begin study of the motion of a falling object. A famous law from physics, known as *Newton's Second Law of Motion*, says that the sum of the forces acting on an object completely determines the acceleration of the object. Precisely:

$$\sum(\text{forces acting on an object}) = (\text{mass of object}) \cdot (\text{acceleration of object})$$

In physics, *vectors* are commonly used to illustrate the forces acting on an object; the resulting picture is called a *free-body diagram* (FBD).

For example, the object shown below has three forces acting on it. If this object is viewed as a *falling* object, then these forces can be interpreted: the force acting down is the force due to gravity; the small force acting upwards is air resistance; and the remaining force could be due to a wind current.



acceleration due to gravity;
$g \approx 32.2$ ft/sec^2

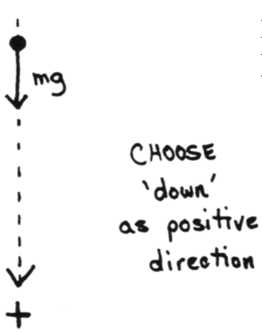
If air resistance and other minor forces are neglected, then the only force acting on a falling body is the force due to gravity. For a particle falling relatively close to the earth's surface, the force due to gravity is given by

$$\text{force due to gravity} = (\text{mass of object})(g) \ ,$$

where g denotes the acceleration due to gravity: $g \approx 32.2$ ft/sec^2

What shall we call the 'positive' vertical direction?

Initially, we'll agree that 'down' is the positive direction.



Newton's Second Law is used to analyze the motion of a falling object; an object traveling along the vertical line shown. First, however, an agreement must be reached about what is the 'positive' direction of this vertical line. Very often, 'up' is considered the positive direction. However, when working with a falling object (which will be traveling *down*), it is often more convenient to decide that 'down' will be the positive direction. Either way will work, as long as one is consistent. Here, we will choose 'down' to be the positive direction. In the exercises, you will get a chance to re-do this example, with 'up' being the positive direction.

using Newton's Second Law

Letting m denote the mass of the falling object, and letting $a(t)$ denote its acceleration at time t, an application of Newton's Second Law (with 'down' the positive direction) says that:

$$\sum \text{forces acting on object} = (\text{mass of object}) \cdot (\text{acceleration of object})$$

That is:

$$mg = m \cdot a(t)$$

Observe that the simplifying assumptions have resulted in only one force acting on the object. This is assumed to be the only force acting on the object throughout its entire fall (until it hits the earth). Note that the force mg appears in this equation as a *positive* constant; this is because the force mg points *down*, and it has been agreed upon that 'down' is the positive direction.

correct sign for $a(t)$ is DETERMINED by the equation

In this equation, the resulting acceleration $a(t)$ of the object is the 'unknown'. The correct sign for $a(t)$ is *determined* by the equation, based on the forces present. That is, the unknown acceleration always appears simply as '$a(t)$'; it would never enter the equation as, say, '$-a(t)$'.

using antidifferentiation to find $v(t)$ and $d(t)$

Once $a(t)$ is found, this information (together with some additional information, to be discussed momentarily) can be used to determine the velocity and distance functions for the particle, by antidifferentiating. How? Well, the falling object has some distance function d that describes its motion along the vertical line; and it must be that $d''(t) = a(t)$. Roughly, we will 'undo' the known derivative $d''(t) = a(t)$ to get information about d' and d.

mass cancels out of the equation of motion

Observe that in the equation $mg = m \cdot a(t)$, the mass cancels out. Thus, under the simplifying assumptions, the resulting acceleration of the object is NOT dependent on the mass of the object. This has an extremely important physical interpretation: if you simultaneously drop a penny and a concrete block from the top of a tall building, they will both hit the ground at the same time!

After cancellation of m, the resulting equation is $g = a(t)$. Since the unknown is $a(t)$, and it is common to put the unknown on the left, the equation is rewritten as $a(t) = g$. Remember that g is a *constant*, $g \approx 32 \text{ ft/sec}^2$.

finding $v(t)$

Since $a(t) = v'(t)$, the equation $a(t) = g$ can be rewritten as:

$$v'(t) = g$$

The unknown velocity function v has derivative g. Do we know ANY function of t that has derivative g? Of course: $y = gt$ has derivative g. Thus, ANY OTHER function with derivative g must have exactly the same shape, but may be translated vertically. That is, any function with derivative g must be of the form $gt + C$ for some constant C.

These thoughts are commonly written down as a list of implications:

$$\begin{aligned} v'(t) = g \quad &\implies \quad \int v'(t)\,dt = \int g\,dt \\ &\implies \quad v(t) = gt + C \end{aligned}$$

There are two important things to note about this mathematical sentence:

two constants of integration have been combined

- When the integration was performed, *two* constants of integration were really obtained: one from the integral on the left, and one from the integral on the right. However, these two constants were combined into a single constant, called C.

$A \implies B \implies C$ means $A \Rightarrow B$ and $B \Rightarrow C$

- This mathematical sentence is of the form '$A \Rightarrow B \Rightarrow C$', which is shorthand for '$A \Rightarrow B$ and $B \Rightarrow C$'. So whenever A is true, then B must be true. And whenever B is true, then C must be true. It follows that whenever A is true, C must be true.

  Letting:

$$\begin{aligned} &A \text{ be the sentence } v'(t) = g \\ &B \text{ be the sentence } \int v'(t)\,dt = \int g\,dt \\ &C \text{ be the sentence } v(t) = gt + C \ , \end{aligned}$$

  we conclude that whenever $v'(t) = g$, then $v(t) = gt + C$ for some constant C.

interpreting the constant of integration; initial velocity, v_0

Read 'v_0' as 'v naught'

Let's investigate the resulting equation $v(t) = gt + C$. This equation gives ALL functions that have derivative g. At time zero, $v(0) = g \cdot 0 + C = C$. Thus, the constant C represents the *initial velocity of the object*, and is commonly denoted by v_0. Read 'v_0' as 'v naught'. Thus, if the *initial velocity* of the falling object is known, then the velocity of the object at ALL times t is known (until some other force enters the picture, like the ground). If the object starts from rest, then the initial velocity is zero.

integrate once more to find $d(t)$

Now, use the fact that $v(t) = d'(t)$, and integrate again:

$$\begin{aligned} v(t) = gt + v_0 \quad &\implies \quad d'(t) = gt + v_0 \\ &\implies \quad \int d'(t)\,dt = \int (gt + v_0)\,dt \\ &\implies \quad d(t) = g \cdot \frac{t^2}{2} + v_0 t + K \end{aligned}$$

initial position, d_0

Read 'd_0' as 'd naught'

At time zero, $d(0) = g \cdot 0 + v_0 \cdot 0 + K = K$, so the constant K represents the *initial position* of the falling object. This initial distance is commonly denoted by d_0. Read 'd_0' as 'd naught'.

choosing the zero reference point on the vertical line

To measure distance along a vertical line, one MUST know where the number '0' lies. There are two common choices: the reference point '0' can coincide with the initial position of the falling object; or, '0' can coincide with the ground. Either choice is fine, providing one remains consistent when interpreting the results. This should become clear in the examples below.

summary

In summary, it has been found that if an object is acted on only by gravity, then its distance function d is given by

$$d(t) = \frac{gt^2}{2} + v_0 t + d_0 \ ,$$

where v_0 represents the initial velocity of the object, and d_0 represents the initial position of the object.

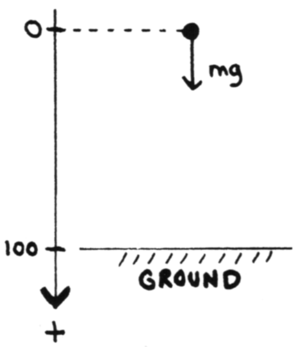
This equation was derived under the assumption that the positive direction of the vertical line is 'down'. The equation is valid until forces other than gravity (like the ground) act on the object.

EXAMPLE

Problem: Suppose that an object is dropped from a height of 100 feet. Answer the following questions:

- What is its distance function?
- How long does it take the object to hit the ground?
- What is the speed of the object when it hits the ground?

Solution #1



Solution #1. It is usually safest to re-derive the equations yourself. It doesn't take very long, and this way you are CERTAIN of the conventions about what is the positive direction, and what is the initial position.

Make a sketch, clearly showing the initial position of the object and the ground. Show the initial force acting on the object. On a vertical line, clearly label your choice for the positive direction, and your choice for '0'. Here, 'down' has been chosen as the positive direction, and '0' coincides with the initial position of the object.

Observe that with this choice of measuring scale, $d(0) = 0$.

$$\begin{aligned}
\overbrace{mg = m\cdot a(t)}^{\text{Newton's second law}} \quad &\Longrightarrow \quad v'(t) = g && \text{(cancel } m,\ a(t) = v'(t)\text{, switch sides)}\\
&\Longrightarrow \quad v(t) = gt + v_0 && \text{(integrate, } v(0) = v_0)\\
&\Longrightarrow \quad v(t) = gt && (v_0 = 0)\\
&\Longrightarrow \quad d'(t) = gt && (v(t) = d'(t))\\
&\Longrightarrow \quad d(t) = \frac{gt^2}{2} + d_0 && \text{(integrate, } d(0) = d_0)\\
&\Longrightarrow \quad d(t) = \frac{gt^2}{2} && (d_0 = 0)
\end{aligned}$$

Thus, the distance function is:

$$d(t) = \frac{gt^2}{2}$$

For the chosen measuring scale, the ground is at position $+100$. So to answer the question: '*How long does it take the object to hit the ground?*', the distance function is set to 100, and solved for t:

$$\begin{aligned}
\frac{gt^2}{2} = 100 \quad &\iff \quad t^2 = \frac{200}{g}\\
&\iff \quad t = \pm\sqrt{\frac{200}{g}}
\end{aligned}$$

The nonnegative number t that makes this true is:

$$t = \sqrt{\frac{200}{g}} \approx \sqrt{\frac{200 \text{ ft}}{32 \text{ ft/sec}^2}} = 2.5 \text{ seconds}$$

The object will hit the ground in approximately 2.5 seconds.

What does $A = B \approx C = D$ mean to us?

Let's be sure we agree upon what the sentence

$$t = \sqrt{\frac{200}{g}} \approx \sqrt{\frac{200\text{ ft}}{32\text{ ft/sec}^2}} = 2.5\text{ seconds}$$

really means. Earlier in the text, it was decided that when a 'chain' like

$$A = B \approx C = D$$

appears, the symbols (in this case, '$\approx$' and '$=$') always compare the objects to their immediate left and right.
Thus,

$$t = \sqrt{\frac{200}{g}}$$

is a true equality, because $\sqrt{\frac{200}{g}}$ is the exact desired time. However,

$$\sqrt{\frac{200}{g}} \approx \sqrt{\frac{200\text{ ft}}{32\text{ ft/sec}^2}}$$

is an approximation, because the value of g is being approximated. And,

$$\sqrt{\frac{200\text{ ft}}{32\text{ ft/sec}^2}} = 2.5\text{ seconds}$$

is a true equality, because $\sqrt{\frac{200}{32}}$ is precisely 2.5.
Note that *if there is at least one '$\approx$' in a chain*, then the first thing in the chain is only approximately equal to the last in the chain. That is, in a chain like

$$A = B \approx C = D\ ,$$

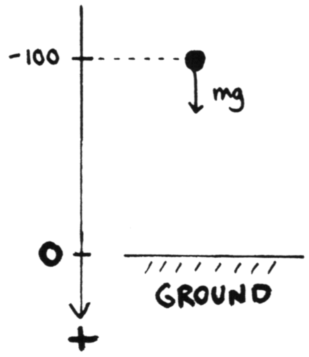
it follows that $A \approx D$. The 'strength' of a chain is determined by its weakest link!

The velocity function was found above to be $v(t) = gt$. Thus, the velocity at time $t = 2.5$ is:

$$v(2.5) = g \cdot (2.5) \approx (32\frac{\text{ft}}{\text{sec}^2})(2.5\text{ sec}) = 80\text{ ft/sec}$$

Observe that the parentheses in $v(2.5)$ are being used for *function evaluation*, NOT multiplication. That is, $v(2.5)$ means the function v, evaluated at 2.5.

Solution #2



Solution #2. This time, a different choice for ‘ 0 ’ is made; ‘0’ coincides with the ground. Since ‘down’ is still the positive direction, the choices lead to $d(0) = -100$. Now we get:

$$\begin{aligned} mg = m \cdot a(t) \quad &\implies \quad v'(t) = g \\ &\implies \quad v(t) = gt + v_0 \\ &\implies \quad v(t) = gt \\ &\implies \quad d'(t) = gt \\ &\implies \quad d(t) = \frac{gt^2}{2} + d_0 \\ &\implies \quad d(t) = \frac{gt^2}{2} - 100 \end{aligned}$$

♣ Fill in a reason for each step in the preceding derivation.

This time, the distance function looks slightly different; it is given by:

$$d(t) = \frac{gt^2}{2} - 100$$

However, *we will obtain precisely the same information* as we did previously. (We must!)

The object hits the ground at time t for which $d(t) = 0$. That is:

$$\frac{gt^2}{2} - 100 = 0$$

This happens when $t = \sqrt{\frac{200}{g}} \approx 2.5$ seconds.

The velocity function is still $v(t) = gt$, so still $v(2.5) \approx 80$ ft/sec.

EXERCISE 6

♣ Re-do the previous example, with the conventions:

- ‘up’ is the positive direction
- ‘0’ coincides with the initial position of the object

Be sure that you obtain the same answers!

EXERCISE 7

♣ Re-do the previous example, with the conventions:

- ‘up’ is the positive direction
- ‘0’ coincides with the ground

Be sure that you obtain the same answers!

EXERCISE 8

Suppose an object is dropped from rest at a height of 200 feet. Answer the following questions, being careful to distinguish '=' from '≈' in your solutions:

♣ 1. What is the distance function for the falling object? What conventions have you used in your derivation?

♣ 2. How long will it take the object to hit the ground?

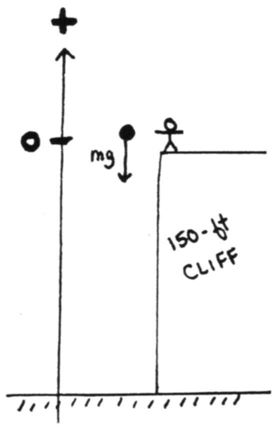
♣ 3. Where is the object after 1 second? 2 seconds?

♣ 4. The object falls past a 100 foot building. How long does it take to reach the top of this building?

♣ 5. What is the velocity of the object at 1 second? 2 seconds? When it hits the ground?

♣ 6. For how many seconds is the equation of motion that you derived valid?

EXAMPLE



Problem: Suppose a person standing at the top of a 150 foot cliff reaches out and throws an object upwards with an initial speed of 10 ft/sec. Answer the following questions:

- What is the distance function for the object? (Derive it.)
- What is the velocity function for the object?
- How long will it go up, before it starts to come down again?
- What is the maximum height that the object will reach?
- How long will it be before the object passes the person who threw it?
- When will the object hit the ground?

Solution. Choose 'up' to be the positive direction, and '0' to coincide with the initial position of the object. Observe that the force acting on the object points DOWN, which is now the negative direction. Then:

$$\begin{aligned}
-mg = m \cdot a(t) \quad &\implies \quad a(t) = -g \\
&\implies \quad v'(t) = -g \\
&\implies \quad v(t) = -gt + v_0 \\
&\implies \quad v(t) = -gt + 10 \\
&\implies \quad d'(t) = -gt + 10 \\
&\implies \quad d(t) = -\frac{gt^2}{2} + 10t + d_0 \\
&\implies \quad d(t) = -\frac{gt^2}{2} + 10t
\end{aligned}$$

Thus, the distance and velocity functions are given by:

$$d(t) = -\frac{gt^2}{2} + 10t \quad \text{and} \quad v(t) = -gt + 10$$

♣ Fill in reasons justifying each step in the preceding derivation.

When the object reaches its maximum height, its velocity is 0:

$$0 = -gt + 10 \quad \iff \quad t = \frac{10}{g} \approx 0.31 \text{ seconds}$$

Thus, the object rises for about 0.31 seconds, before it turns around to come down again.

At $t = 0.31$:

$$d(0.31) = -\frac{g(0.31)^2}{2} + 100(.31) \approx 1.56 \text{ feet}$$

Remember that this position is relative to the '0' mark; the top of the cliff. Thus, the maximum height the object reaches is $150 + 1.56 = 151.56$ feet. (Observe that the height of the person who threw the object is being neglected.)

The person on the cliff is at position 0 relative to the chosen scale. Thus, we must set $d(t)$ equal to 0 and find the nonnegative value of t that makes this true:

$$\begin{aligned} -\frac{gt^2}{2} + 10t = 0 \quad &\iff \quad t\left(-\frac{gt}{2} + 10\right) = 0 \\ &\iff \quad t = 0 \ \text{ or } \ -\frac{gt}{2} + 10 = 0 \\ &\iff \quad t = 0 \ \text{ or } \ t = \frac{20}{g} \approx 0.63 \text{ seconds} \end{aligned}$$

It takes the object about 0.63 seconds to pass the person who threw it.

The object hits the ground when $d(t) = -150$, relative to the chosen scale:

$$\begin{aligned} -\frac{gt^2}{2} + 10t = -150 \quad &\iff \quad -\frac{gt^2}{2} + 10t + 150 = 0 \\ &\iff \quad t = \frac{-10 \pm \sqrt{(10)^2 - 4(-\frac{g}{2})(150)}}{2(-g/2)} \\ &\iff \quad t \approx -2.77 \text{ secs} \ \text{ or } \ t \approx 3.39 \text{ secs} \end{aligned}$$

Choosing the nonnegative answer, the object hits the ground after approximately 3.39 seconds.

QUICK QUIZ
sample questions

1. What is the difference between 'speed' and 'velocity'?
2. Suppose that the distance function for an object is given by $d(t) = t^2 + 2t$. Let distance be measured in feet, time in seconds. Find the position, velocity, speed, and acceleration of the object at $t = 1$.
3. What is a 'vector'?
4. What is a 'free body diagram'?
5. Suppose that $v(t) = gt$. In the sentence '$v(2) = g(2)$', what does '$v(2)$' mean? What does '$g(2)$' mean?

KEYWORDS
for this section

Motion along a line, instantaneous velocity and acceleration, velocity versus speed, vectors, vectors in a line, vectors in space, free-body diagrams, acceleration due to gravity, Newton's second law of motion, using antidifferentiation to find $v(t)$ and $d(t)$, interpreting the constants of integration, distinguishing between '$=$' and '$\approx$'.

END-OF-SECTION EXERCISES

♣ Suppose a person standing at the top of a 75 foot cliff reaches out and throws an object upwards with an initial speed of 20 ft/sec. You may ignore the height of the person throwing the object. Answer the following questions:

1. What is the distance function for the object? (Derive it. Use any appropriate conventions.)
2. What is the velocity function for the object?
3. How long will it go up, before it starts to come down again?
4. What is the maximum height that the object will reach?
5. How long will it be before the object passes the person who threw it?
6. When will the object hit the ground?