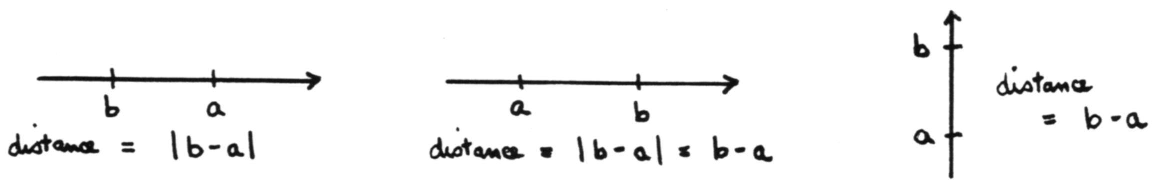
7.5 The Area Between Two Curves

Introduction

The definition of the definite integral provides mathematicians with *intuition* that helps to develop formulas involving the definite integral. First, a formula is developed for finding the area of the region between two curves.

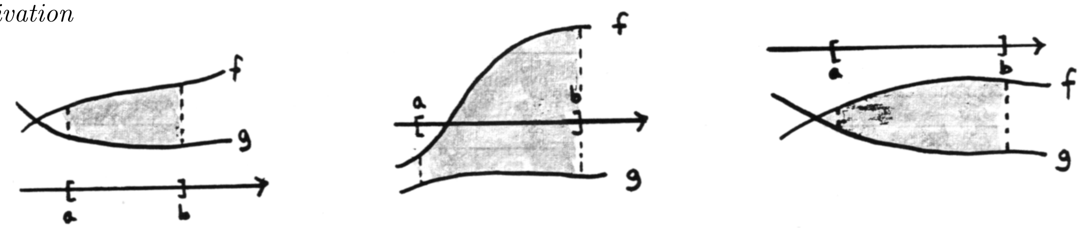
distance between two real numbers

Recall that if a and b are *any* two real numbers, then the distance between them is $|b-a|$. If, in addition, $b \geq a$, then the distance between them is $|b-a| = b-a$.



finding the area between two curves; a motivation

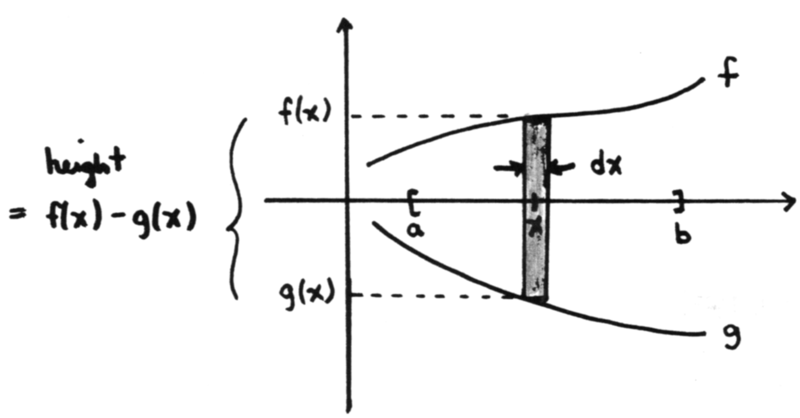
Let f and g both be continuous on the interval $[a,b]$, and suppose that $f(x) \geq g(x)$ for all $x \in [a,b]$, so that the graph of f lies above the graph of g on $[a,b]$.



We want to find the area between the graphs of f and g on $[a,b]$. To motivate the formula, proceed as follows:

Investigate a typical 'infinitesimal slice' of the desired area. First, choose a value of x between a and b, and look at a slice of the desired area at this value x. Denote the width of this typical slice by dx (think of dx as denoting an infinitesimal piece of the x-axis).

Since $f(x) \geq g(x)$, the height of the slice is $f(x) - g(x)$. Observe that this is the height of the slice, *regardless* of the signs (plus or minus) of f and g.



Therefore, the area of this typical slice is:

$$\overbrace{\big(f(x) - g(x)\big)}^{\text{height}} \ \overbrace{dx}^{\text{width}}$$

Now, use calculus to 'sum' these slices:

$$\text{desired area} = \int_a^b \big(f(x) - g(x)\big)\ dx$$

Although this is certainly not a rigorous development of the formula (which would require partitioning $[a, b]$ and investigating Riemann sums), the result is correct. The process illustrates how intuition about the definite integral can be used to gain some useful results.

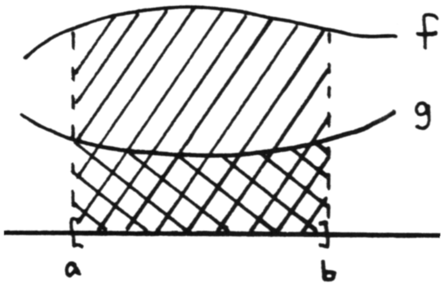
EXERCISE 1

♣ 1. Show that whenever $f(x) \geq g(x)$, then $f(x) - g(x) \geq 0$. Be sure to write a complete mathematical sentence. (This is a one-liner.)

♣ 2. Suppose that $f(1) = -2$ and $g(1) = -4$. Plot the two points described here. Is $f(1) \geq g(1)$? What is $f(1) - g(1)$ in this case?

another viewpoint

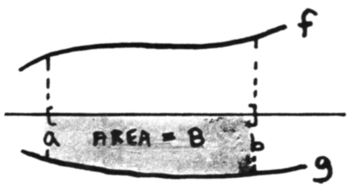
Now, view the previous problem from a different perspective. Suppose for the moment that *both* f and g are positive, and the graph of f lies above the graph of g. Then, the area between f and g can be found by finding the area under f, and subtracting off the area beneath g:



$$\begin{aligned}\text{area between } f \text{ and } g &= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \\ &= \int_a^b \big(f(x) - g(x)\big)\,dx \quad \text{(by linearity)}\end{aligned}$$

Similarly, if f is positive and g is negative, then the graph of f necessarily lies above the graph of g, and the desired area can be found as follows:

Keep in mind that the word 'area' always refers to a *nonnegative* quantity.



In this situation (illustrated in the sketch), $\int_a^b g(x)\,dx$ is a *negative* number, since the definite integral treats area beneath the x-axis as negative. Thus, $B = -\int_a^b g(x)\,dx$. The desired area between the two curves on $[a, b]$ is then:

$$\begin{aligned}\text{desired area} &= \int_a^b f(x)\,dx + \left(-\int_a^b g(x)\,dx\right) \\ &= \int_a^b \big(f(x) - g(x)\big)\,dx \quad \text{(by linearity)}\end{aligned}$$

The same formula is again obtained.

EXERCISE 2

♣ Suppose that f and g are *both negative* on the interval $[a, b]$, and that $f(x) \ge g(x)$ on $[a, b]$. Make a sketch that illustrates this situation. Then, proceeding as in the previous example, find the formula for the area between f and g on $[a, b]$.

The result concerning the area between two curves is summarized below:

AREA BETWEEN TWO CURVES

Let f and g be continuous on $[a, b]$, and suppose that $f(x) \ge g(x)$ on $[a, b]$, so that the graph of f lies above the graph of g. Then, the area between the graphs of f and g on $[a, b]$ is given by:

$$\int_a^b \big(f(x) - g(x)\big)\,dx$$

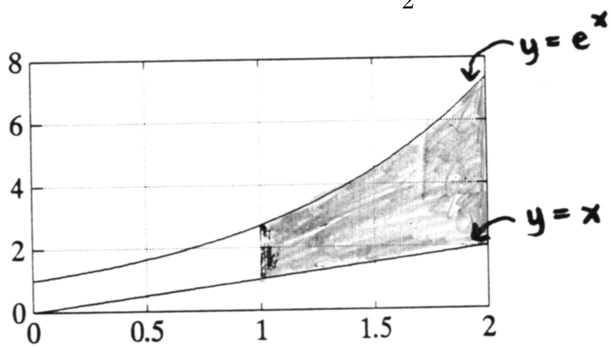
EXAMPLE

finding the area between two curves

Problem: Find the area between $y = e^x$ and $y = x$ on $[1, 2]$.

Solution: A quick sketch shows that the graph of $y = e^x$ lies above the graph of $y = x$ on the interval $[1, 2]$. Thus, the desired area is given by:

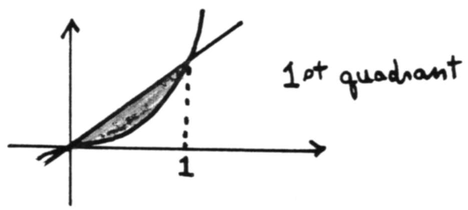
$$\begin{aligned}\int_1^2 (e^x - x)\,dx &= (e^x - \frac{x^2}{2})\Big|_1^2 \\ &= (e^2 - 2) - (e - \frac{1}{2}) \\ &= e^2 - e - \frac{3}{2} \approx 3.171\end{aligned}$$



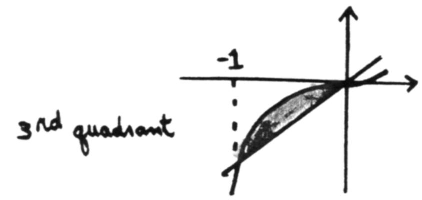
the phrase 'bounded by'

The phrase '*bounded by ...*' can be roughly interpreted as '*having edges (boundary) given by the graphs of ...*'. The idea is illustrated in the following examples:

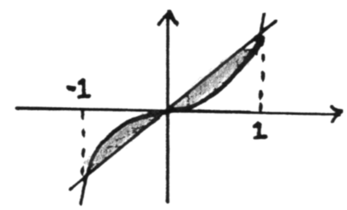
- The area in the first quadrant bounded by $y = x$ and $y = x^3$ is shown below. This is the area in the first quadrant that has as its boundary *only* the graphs of $y = x$ and $y = x^3$.

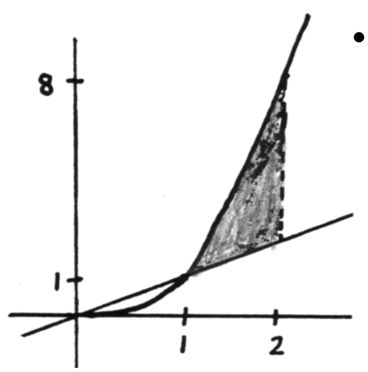


- The area in the third quadrant bounded by $y = x$ and $y = x^3$ is shown below. This is the area in the third quadrant that has as its boundary *only* the graphs of $y = x$ and $y = x^3$.

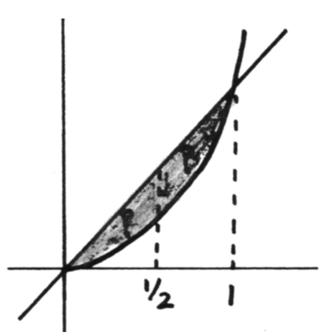


- The area bounded by $y = x$ and $y = x^3$ is shown below. This is the area that has as its boundary the graphs of $y = x$ and $y = x^3$. Observe that this area is naturally composed of two pieces.



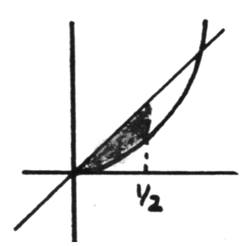


- The area bounded by $y = x$, $y = x^3$, and $x = 2$ is shown at left. This is the area having as its boundary the graphs of the given equations. Remember that the graph of $x = 2$ (viewed as an equation in two variables) is the set of all points (x, y) with $x = 2$. That is, the graph of $x = 2$ consists of all points with x-value equal to 2; thus, it is the vertical line that crosses the x-axis at 2.



- The phrase 'the area bounded by $y = x$, $y = x^3$ and $x = \frac{1}{2}$' is *ambiguous*; there are two adjacent pieces of area with the given edges. Do we want just one of these? Both of these? If both are desired, why wasn't the simpler description 'the area in the first quadrant bounded by $y = x$ and $y = x^3$' given?

  Because of this ambiguity, the desired area is clarified by using, say, the description 'the area bounded by $y = x$, $y = x^3$, $x = 0$, and $x = \frac{1}{2}$'. Alternately, the description 'the area between $y = x$ and $y = x^3$ on $[0, \frac{1}{2}]$' can be used. In either case, there is no doubt that the area being described is the one shown below.



EXERCISE 3 Sketch each of the areas described below:

♣ 1. The area bounded by $y = x^2$ and $y = x$.

♣ 2. The area in the first quadrant bounded by $y = x^2$ and $y = x^4$.

♣ 3. The area in the second quadrant bounded by $y = x^2$ and $y = x^4$.

♣ 4. The area bounded by $y = x^2$ and $y = x^4$.

♣ 5. The area bounded by $y = x^2$, the x-axis, $x = 1$ and $x = 3$.

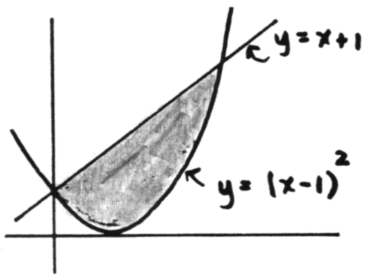
♣ 6. The area bounded by $y = x^2$, $y = -x^2$, $x = 1$ and $x = 3$.

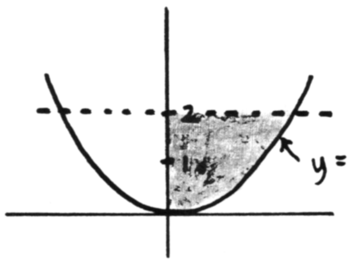
♣ 7. The area bounded by $y = x^2$, $y = 1$, and $y = 2$.

♣ 8. The area bounded by $y = x^2$ and $y = 4x - 3$.

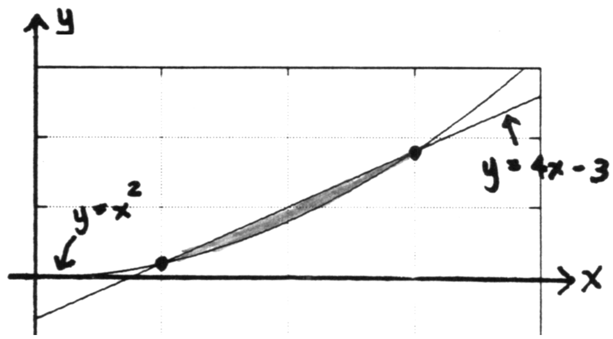
EXERCISE 4 ♣ Describe each of the areas shown below, using an appropriate variation of the phrase:

'*the area bounded by* . . . '





EXAMPLE Problem: Find the area bounded by $y = x^2$ and $y = 4x - 3$.



finding intersection points

Solution: It is first necessary to find the intersection points. To do this, we seek points (x, y) that make *both* equations true (so that the point (x, y) lies on *both* curves).

If (x, y) is an intersection point, then when the number x is substituted into *either* equation, the *same* value of y results. Therefore, values of x are sought for which the y values on *both curves* are the same. To find such values, set the y values of both curves equal to each other, and solve for the corresponding value(s) of x:

$$\begin{aligned} x^2 = 4x - 3 &\iff x^2 - 4x + 3 = 0 \\ &\iff (x-1)(x-3) = 0 \\ &\iff x = 1 \ \text{ or } \ x = 3 \end{aligned}$$

EXERCISE 5

review of equivalence and the mathematical words 'or' and 'and'

♣ 1. In English, what does the sentence '$x^2 = 4x - 3 \iff x = 1$ or $x = 3$' mean?

♣ 2. For what value(s) of x is the sentence '$x = 1$ or $x = 3$' true? (If necessary, review the mathematical meaning of the word 'or'.)

♣ 3. Suppose that, for a given value of x, the sentence '$x^2 = 4x - 3$' is false. Can anything be said about the truth value of the sentence '$x = 1$ or $x = 3$'?

♣ 4. Is it correct to say

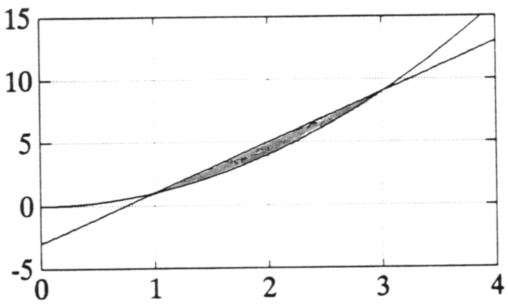
$$(x-1)(x-3) = 0 \iff x = 1 \ \text{ and } \ x = 3 \ ?$$

Why or why not? (If necessary, review the mathematical meaning of the word 'and'.)

So the curves intersect when $x = 1$ and when $x = 3$. Since the points with these x values lie on *both* curves, *either* curve, $y = x^2$ or $y = 4x - 3$, can be used to find the corresponding y-values:

$$\begin{aligned} & x = 1 \implies y = 1^2 = 1 && \text{(substituting into } y = x^2) \\ \text{or} \quad & x = 1 \implies y = 4(1) - 3 = 1 && \text{(substituting into } y = 4x - 3) \end{aligned}$$

$$\begin{aligned} & x = 3 \implies y = 3^2 = 9 && \text{(substituting into } y = x^2) \\ \text{or} \quad & x = 3 \implies y = 4(3) - 3 = 9 && \text{(substituting into } y = 4x - 3) \end{aligned}$$



use the simplest curve to find the corresponding y-values

Since *either* curve can be used to find the corresponding y-values, one usually chooses the simplest one. (In this example, it would be a toss-up as to which curve is simpler.)

From the sketch, the graph of $y = 4x - 3$ lies above the graph of $y = x^2$ on $[1, 3]$. (Momentarily, it will be observed that it is not really necessary to know which curve is on top.) Thus, the desired area is given by:

$$\begin{aligned}\int_1^3 \left((4x-3) - x^2\right) dx &= 2x^2 - 3x - \frac{1}{3}x^3\Big|_1^3 \\ &= (18 - 9 - 9) - (2 - 3 - \frac{1}{3}) \\ &= \frac{4}{3}\end{aligned}$$

EXAMPLE

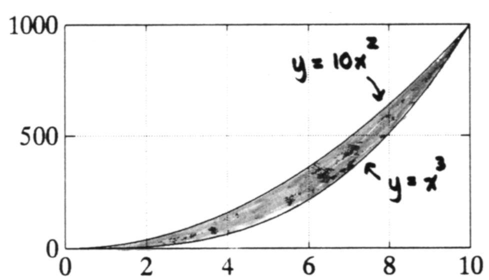
it's not really necessary to know which curve is on top

Problem: Find the area in the first quadrant bounded by $y = 10x^2$ and $y = x^3$.

Solution: It's not necessary to graph the functions; just find the intersection points:

$$\begin{aligned}10x^2 = x^3 &\iff x^3 - 10x^2 = 0 \\ &\iff x^2(x - 10) = 0 \\ &\iff x = 0 \text{ or } x = 10\end{aligned}$$

The curves intersect at $x = 0$ and $x = 10$. So, on the interval $[0, 10]$, either the graph of x^3 is on top, or the graph of $10x^2$ is on top. (If one were on top for only *part* of the time, then there would have to be another intersection point.) Just *guess* that x^3 is on top, and calculate:



$$\begin{aligned}\int_0^{10} \left(x^3 - 10x^2\right) dx &= \frac{1}{4}x^4 - \frac{10}{3}x^3\Big|_0^{10} \\ &= (\frac{10000}{4} - \frac{10000}{3}) \\ &= 10000(\frac{3}{12} - \frac{4}{12}) \\ &= -\frac{2500}{3}\end{aligned}$$

Since the answer is negative, the guess was incorrect: actually, $y = 10x^2$ is on top. But the desired area is still known:

$$\text{desired area} = \left|-\frac{2500}{3}\right| = \frac{2500}{3}$$

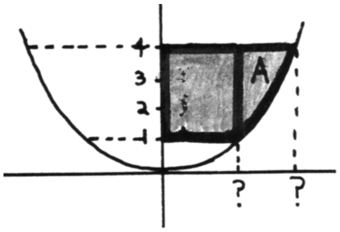
EXAMPLE

jigsaw puzzles

Problem: Find the area bounded by $y = x^2$, $y = 1$ and $y = 4$.

Solution: If you like jigsaw puzzles, you'll like this sort of problem. Just figure out the easiest 'pieces' that make up the desired area.

Symmetry can be used to simplify the problem. Find the area in the first quadrant, and double it.



One way to view the area in the first quadrant is as being composed of a rectangle, and an extra piece.

The height of the rectangle is $4 - 1 = 3$. What is the width? To answer this, we need to know where $y = 1$ and $y = x^2$ intersect:

$$x^2 = 1 \iff x = \pm 1$$

(Remember that '$x = \pm 1$' is shorthand for '$x = 1$ or $x = -1$'.)

Thus, the width of the rectangle is $1 - 0 = 1$. The rectangle has area $(3)(1) = 3$.

What is the area of the remaining piece? We need to know where $y = 4$ and $y = x^2$ intersect:

$$x^2 = 4 \iff x = \pm 2$$

Then, calling the area of this piece A:

$$\begin{aligned} A = \int_1^2 (4 - x^2)\,dx &= 4x - \frac{x^3}{3}\Big|_1^2 \\ &= (8 - \frac{8}{3}) - (4 - \frac{1}{3}) \\ &= 4 - \frac{7}{3} = \frac{12}{3} - \frac{7}{3} = \frac{5}{3} = 1\frac{2}{3} \end{aligned}$$

Therefore, the area bounded by $y = x^2$, $y = 1$ and $y = 4$ is:

$$2 \cdot (3 + 1\frac{2}{3}) = 9\frac{1}{3}$$

QUICK QUIZ
sample questions

1. Suppose that g and f are continuous functions, and that $g(x) \ge f(x)$ on the interval $[c, d]$. Give a formula for the area between f and g on $[c, d]$.
2. Find the area bounded by $y = -x^2 + 1$ and the x-axis.
3. Is the phrase 'the area bounded by $y = (x - 2)^2$, $x = 1$ and $y = 4$' ambiguous? Why or why not?
4. Find the area between $y = e^x$ and $f(x) = -x$ on $[0, 1]$. Make a sketch showing the area that you are finding.

KEYWORDS
for this section

The distance between two real numbers, finding the area between two curves, the phrase 'bounded by', finding intersection points, it is not necessary to know which curve is on top.

END-OF-SECTION EXERCISES

♣ Find the area of each region described below. Make a sketch, and shade the area that you are finding. Be sure to write complete mathematical sentences.

1. In the first quadrant, bounded by: $y = x^2$ and $y = x^4$
2. In the third quadrant, bounded by: $y = x$ and $y = x^3$
3. Bounded by $y = -(x - 2)^2 + 3$ and $y = -1$
4. Bounded by $y = x$ and $y = x^3$
5. Bounded by $y = x^2$, $y = 1$ and $y = 2$
6. Bounded by $y = x^2$ and $y = -1$, $x = 0$ and $x = 2$
7. Bounded by $y = x^3$, $y = 8$ and $x = -1$